Optimal Brand Umbrella Size

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March 2001

Abstract

In a framework or repeated-purchase experience goods with seller’s moral hazard, umbrella branding may improve the terms of the “implicit contract” between firm and consumers, whereby the firm invests in quality and consumers pay a high price. The reason is that the costs from cheating consumers are greater under umbrella branding (loss of reputation in all products), which implies that the high-quality equilibrium is more stable. The reverse side of this coin is that, if reputation breaks down in one product (which is a distinct possibility in a world with imperfect observability), then the bad news travel faster under umbrella branding.

Putting these two considerations together, I show that umbrella branding is a superior strategy when there is a significant overlap between the set of buyers of each of the firm’s products. This result extends the well-known notion that brand extensions and umbrella branding are only successful if there is a good “fit” between the different products under the same umbrella. Specifically, my result points out that such fit need not necessarily correspond to objective product characteristics but rather to the identity of buyers.

*I am grateful to participants at the Berkeley marketing seminar for useful comments and suggestions. The useful disclaimer applies.
1 Introduction

Umbrella branding, the practice of labeling more than one product with a single brand name, is common practice among multiproduct firms in a variety of markets. Examples include Canon cameras and photocopiers, Colgate toothpaste and toothbrushes, Levi’s jeans and sneakers. To underscore the importance of the strategy of umbrella branding, Aaker and Keller (1990) quote a Nielsen report according to which “from 1977 to 1984, approximately 40% of the 120 to 175 new brands that were introduced into supermarkets annually were extensions” (p. 27).

In this paper, I propose a model that highlights the costs and benefits of umbrella branding and ultimately provides the basis for a theory of optimal umbrella size. I consider a framework or repeated-purchase experience goods with seller’s moral hazard. Firms have a short-run incentive to reduce quality and save costs, as consumers can only observe quality ex post. However, there exist equilibria whereby firms refrain from cheating consumers. In these equilibria, when consumers infer that the firm has cheated them the firm’s reputation breaks down, whereby consumers no longer pay a high price for the firm’s product and the firm no longer produces quality products.

I show that umbrella branding may improve the terms of the “implicit contract” between firm and consumers. The reason is that the costs from cheating consumers are greater under umbrella branding (loss of reputation in all products), which implies that the high-quality equilibrium is more stable. The reverse side of this coin is that, if reputation breaks down in one product (which is a distinct possibility in a world with imperfect observability), then the bad news travel faster under umbrella branding.

Putting these two considerations together, I show that umbrella branding is a superior strategy when there is a significant overlap between the set of buyers of each of the firm’s products. This result extends the well-known notion that brand extensions and umbrella branding are only successful if there is a good “fit” between the different products under the same umbrella. Specifically, my result points out that such fit need not necessarily

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1 The terms “brand extension” or “brand stretching” are also used. Some authors distinguish between “line extension” (when the new product is in the same class) and “brand extension,” when the new product is in a different class. Given the level of generality of the model considered in this paper, this distinction is not crucial.
correspond to objective product characteristics but rather to the identity of buyers.

**Related literature.** There is a fairly sizeable literature on brand extension and umbrella branding. One first explanation for brand extensions is that umbrella branding is a form of economies of scope, as it economizes on the costs of creating a new brand. Tauber (1988), for example, argues that

> World-competitive pressures and slow-growth markets keep cost containment as a primary management imperative. In this environment, growth through brand leverage will continue to flourish (p. 30).

A related idea is that brands have an intrinsic value (status or otherwise). Brands are therefore like a “public good” in the sense that the more products are sold under the same brand the greater the total value created. See Pepall and Richards (2000).

A different perspective on brand extensions is that, in a world where consumers are uncertain about product characteristics (due to horizontal or vertical differentiation), brands may play an informational role. As the *Economist* argues (July 2nd, 1994),

> Brands are created because buyers crave information. They see a huge range of products that look the same and seem to perform similar. Brands offer a route through the confusion.

Wernerfelt (1988) suggests that, if umbrella branding is costly, then it may serve as a signal of new product quality that is used by high-quality firms only. In a similar context (adverse selection), Montgomery and Wernerfelt (1992) show that, in a free-entry equilibrium, single-product and multiple-product (umbrella) brands coexist. Still in a similar context, Cabral (2000) studies the direct and feed-back reputational effects of brand extensions.

In a framework of horizontal product differentiation, Sappington and Wernerfelt (1985) argue that umbrella branding may reduce uncertainty about a new product’s attributes, a fact that increases value if consumers are risk averse. They also present empirical evidence consistent with the predictions of their model.\(^2\)

More closely related to my paper, Choi (1998) proposes a moral-hazard theory of brand extensions. In a repeated-game framework, sellers introduce high-quality products and buyers pay a high price for it. Buyers can observe quality ex post and punish sellers who cheat by

\(^2\)Other papers that include empirical evidence relating to brand extensions include Aaker and Keller (1990), Sullivan (1990) and Erdem (1998).
introducing low-quality products; the nature of the punishment is that future introductions are no longer believed to be of high quality as before. My paper differs from Choi (1998), among other things, in that it considers the costs, not only the benefits, from umbrella branding.\(^3\)

**Coming next.** The plan of the paper is as follows. In Section 2, I lay down the basics of the model and the main assumptions. Section 3 establishes the parallel between the provision of quality in a moral hazard context (as in my model) and oligopoly collusion. This parallel is useful insofar as results from collusion theory can be applied to the problem of product quality and umbrella branding. Some of these results are presented in Section 4, which also includes results specific to the problem of umbrella branding. Section 6 concludes the paper.

### 2 Model

Consider an economy with a continuum of single-product firms and a countable number (measure zero) of two-product firms.\(^4\) In each period, each firm must decide whether or not to spend effort into producing quality products; a product is of high quality if and only if the firm spends effort on it. The cost of effort toward quality is as follows. A one-product seller pays \(\epsilon\) to produce a high-quality product, whereas low-quality products are costless. A two-product seller pays \(\epsilon'\) to produce one high-quality product and \(\epsilon''\) to produce two high-quality products; producing two low-quality products costs zero. I make the following assumption regarding the costs of effort:

**Assumption 1 (economies of scope)** \(\epsilon < \epsilon' < \epsilon'' < 2\epsilon\).

In words, the assumption means that there are economies of scope in providing quality: a two-product firm has a lower cost per high-quality product than a one-product firm.

Consumers cannot observe quality ex ante, only ex post and even so imperfectly; specifically, consumers only observe whether a product performs well or rather breaks down. If the

\(^3\)A different view on the benefits from umbrella branding is given by Choi and Scarpa (1992), who show that brand extensions can allow an incumbent to credibly deter entry.

\(^4\)The assumption that there is only a countable number of two-product firms is not crucial but greatly simplifies the analysis.
product is of low quality, then it breaks down with probability one. High-quality products, in turn, break down with probability $\gamma$. Consumers value high-quality products at $\pi$ and low-quality products at zero.$^5$

The customers of a typical two-product firm can be divided into two categories: type $\alpha$ consumers, who purchase only one of the firm’s products; and type $\beta$ consumers, who purchase both. To simplify the notation, I also denote by $\alpha$ and $\beta$ the fraction of each buyer type.

The focus of the analysis will be on two-product firms. These firms face two main decisions. At time of birth, they must decide whether to sell both products under the same name, or rather under different names. Then, in each period, they must choose the quality level of each of their products.

The firm’s choice of name is important because consumers cannot observe the ownership of each brand:

**Assumption 2 (brand ownership)** Consumers observe brand names but not the identity of each brand owner.

A consequence of Assumption 2 is that consumers cannot observe whether a particular firm sells one or two products except when the firm sells two products under the same name and consumers buy both products (type $\beta$ consumers). Since I assume that there is only a countable number of two-product firms, consumers take every firm to be a one-product firm unless they actually observe two products being sold under the same brand name.

I make two additional assumptions regarding consumer behavior:

**Assumption 3 (bounded recall)** In each period, each consumer only knows the quality of the products purchased in the previous period.

**Assumption 4 (symmetry)** All consumers of the same type behave equally.

In the next section, I discuss the relevance of these assumptions. A summary of the model’s notation can be found in Table 1.

$^5$In other words, if consumers are risk neutral, then they value a functional product at $\pi/\gamma$. 

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Table 1: Notation.

\begin{tabular}{ll}
\hline
$\alpha$ & fraction of consumers who only buy one product \\
$\beta$ & fraction of consumers who buy two products \\
$\gamma$ & probability that a high-quality product breaks down \\
$\delta$ & discount factor \\
$\epsilon$ & cost of quality for a one-product firm \\
$\epsilon'$ & cost of one quality product for a two-product firm \\
$\epsilon''$ & cost of two quality products for a two-product firm \\
$\pi$ & consumer valuation for a high-quality product \\
$\theta$ & probability of reputation breakdown \\
\hline
\end{tabular}

3 Quality provision and oligopoly collusion

Since quality is costly and can only be (imperfectly) observed ex post, no myopic firm would have an incentive to spend effort. In a repeated-purchase context, however, there exist equilibria where firms produce high-quality products. The formal analysis of such equilibria can be found in the pioneering work of Klein and Leffler (1981) and Shapiro (1983).\(^6\) In these papers, quality can be observed perfectly ex post; high-reputation firms spend effort and produce high-quality products; consumers expect high-reputation firms to produce high-quality products and pay a price according to such expectation; finally, if a high-quality firm ever produces a low-quality product then consumers lose trust in such a firm and the firm’s reputation breaks down: from then on, such firm produces low-quality products and consumers pay a price according to this revised expectation. As a corollary of the folk theorem, if firms are sufficiently patient then such a strategy profile is indeed an equilibrium: even though, in the short run, firms would have an incentive to cheat on consumers and produce low quality, such deviation would not pay given the costs from squandering the reputation.

The model I consider is more complicated because quality is only imperfectly observed; the Klein-Leffler-Shapiro results are therefore not immediately applicable. At this point, it might be worth noting that the analysis of the quality model is analogous to the problem of collusion between two firms. In fact, just as the quality model consists of an implicit

\(^6\)See also Telser (1980), Kreps (1990).
“contract” between the firm and consumers, so the problem of duopoly collusion consists of an implicit contract between competitors. This analogy is useful as it justifies the application of the results from oligopoly theory to the problem of quality and reputation.

The problem of oligopoly collusion with imperfect monitoring has been analyzed by Green and Porter (1984) and others. In particular, Tirole (1989) considers a simplified version of the Green-Porter theory whereby each firm can receive low demand either because the rival cuts price or because market demand is low. This element of imperfect monitoring implies a greater incentive for firms to “secretly” cut their prices and increase market share. A collusive agreement is still possible in this case, though its nature is different from the case of perfect monitoring. Green and Porter (1984) and Tirole (1989) show that there exists an optimal equilibrium (that is, one that maximizes firms’ profits) with the following features: firms set high prices; if, in any period, one of the firms receives low demand, then for $T$ periods firms set low prices, upon which they revert to high prices again; $T$ is the minimum value of $T$ such that each firm’s no-deviation constraint is exactly met.

It should be noted that this temporary-price-war equilibrium, though optimal, is not the only optimal equilibrium. Equally optimal would be an equilibrium whereby, upon observing low demand, firms revert to an infinite price war with probability $\theta$, where $\theta$ is the minimum value such that the no-deviation constraint is met. However, given the empirical evidence from industries such as grain shipping (cf Porter, 1983), the temporary-price-war equilibrium seems more realistic.

By analogy with the problem of optimal collusion under imperfect monitoring, we can also find equilibria of the model of quality provision with imperfect observability of quality. The main difference with respect to oligopoly collusion is that, given Assumptions 3 and 4, the equivalent to the temporary-price-war equilibrium cannot be implemented. That is, there can be no equilibrium whereby reputation breaks down during $T$ periods. In fact, as we will see below, an optimal equilibrium has the same features as the collusion equilibrium that features infinite price wars with probability $\theta$. 
4 Main results

In this section, I derive the optimal equilibria in the quality provision game with imperfect observability. To begin, I define optimal equilibria:

**Definition 1 (optimal equilibrium)** *An optimal equilibrium is a subgame perfect Nash equilibrium that maximizes the seller’s expected discounted profit at time zero.*

As mentioned above, the focus of the paper is on two-product firms. However, for completeness and for future reference, it is useful to start with the analysis of a one-product seller.

**One-product firms.** The optimal equilibrium for a one-product seller is conceptually similar to a Green-Porter-Tirole type of equilibrium where firms collude and demand is imperfectly observable. Specifically, in each period the seller invests in quality and consumers pay the amount corresponding to a high-quality product. If, in a given period, the product breaks down, then the firm’s reputation breaks down forever with probability $\mu$. Once reputation breaks down, consumers expect the firm to produce a low quality product and are unwilling to pay any positive price; accordingly, the firm produces low quality as it has no incentive to produce high quality. Finally, the value of $\mu$ is such that the firm is just indifferent between investing and not investing in quality.

Given the assumptions of finite recall and buyer symmetry, this is in fact the unique optimal equilibrium. Formally,

**Proposition 1** *If $\gamma \leq 1 - \frac{\epsilon}{\delta\pi}$, then the optimal equilibrium consists of the following:*

(i) the seller invests in quality until its reputation breaks down;

(ii) consumers purchase the product at a price $\pi$ if reputation is high; if reputation is low, then no consumer buys the product;

(iii) reputation starts at the high level and breaks down (forever) with probability $\theta$ each time the product fails;

(iv) the value of $\theta$ is increasing in $\gamma, \epsilon$; and decreasing in $\delta, \pi$. 

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Two-product firms with type $\beta$ consumers. Let us now consider the case of a firm that sells two products. Specifically, I first consider the extreme case of a firm that sells its two products under the same name and to the same set of consumers; that is, I assume umbrella branding and $\beta = 1$: all consumers are of type $\beta$. An equilibrium now specifies the probability that the firm’s reputation breaks down in case of one or two product failures. Let those be given by $\theta_1$ and $\theta_2$, respectively.

Proposition 2 Consider a two-product firm selling to type $\beta$ consumers only. In an optimal equilibrium, one of the following must be true:

- Failure of one isolated product does not affect the firm’s reputation ($\theta_1 = 0$);
- Failure of both products implies the breakdown of the firm’s reputation with probability one ($\theta_2 = 1$).

In words, the interpretation of Proposition 2 (and its proof) is the following. For high values of the discount factor, $\theta_1 = 0$ and $\theta_2 > 0$. This means that, when one of the two products breaks down, consumers “assume” such event to be the result of “bad luck” and reputation remains intact; when, in turn, two products break down, then the firm’s reputation breaks down as well (with positive probability). For lower values of the discount factor, $\theta_1 > 0$ and $\theta_2 = 1$. Now the failure of two products automatically implies the breakdown of the firm’s reputation, whereas the failure of one product only implies the breakdown of reputation with probability lower than one. For all values of the discount factor, consumers put more weight on the event of the failure of two products than on the event of one product failure only. More generally, Proposition 2 implies that $\theta_1 \leq \theta_2$, that is, one-product failures are treated more leniently than two-product failures.

The intuition for Proposition 2 is that, along the equilibrium path (that is, assuming the seller invests $c''$ to produce two high-quality products), the probability of two product failures is very small, namely $\gamma^2$ (if the value of $\gamma$ is small, then the value of $\gamma^2$ is very small). It is therefore optimal to impose a large “punishment” following the event of two product failures and be very lenient when only one product breaks down. In fact, while two simultaneous product failures rarely occur along the equilibrium path, deviation by the firm implies two product failures with probability $\gamma$ (if it invests $c'$) or 1 (if it invests zero).
Main results: to umbrella brand or not to umbrella brand. Proposition 2 implies that, if all consumers are of type $\beta$, then it is optimal to umbrella brand. In fact, under no umbrella branding the optimal equilibrium consists of applying the results of Proposition 1 to each of the firm’s products. The result is an equilibrium with discounted profit lower than that of Proposition 2. To see that payoff is actually lower, notice that one possible solution under umbrella branding would be precisely to duplicate the solution under no umbrella branding.

By continuity, Proposition 2 implies that, if $\beta$ is high enough, then umbrella branding is a superior strategy:

Proposition 3 There exists a $\overline{\beta}$ such that, if $\beta > \overline{\beta}$, then the optimal equilibrium under umbrella branding is superior to the optimal equilibrium under no umbrella branding.

The above simple model of moral hazard in quality provision thus highlights one important benefit from umbrella branding: it improves the conditions for the “implicit” contract between firm and consumers in the repeated sale of an experience good. In this contract, the firm provides high-quality products and consumers pay a high price. The reason why umbrella branding is welfare enhancing is that the firm’s no-deviation constraints are tighter than under no umbrella branding, thus reducing the incentives for the firm to cheat consumers and assuring a more stable equilibrium. In fact, under umbrella branding a breakdown of reputation implies the loss of revenues from all of the firm’s products, a stronger “punishment” than the loss of reputation in one product only.\footnote{It should be noted that this result only holds under imperfect observability of quality. Under perfect observability, the no-relevance result of Bernheim and Whinston (1990) applies: although the punishment from deviation is twice as large under umbrella branding, so is the benefit from cheating consumers.}

Is it then the case that umbrella branding is a dominant strategy? In the preceding analysis, I have assumed that all consumers are $\beta$ types. As the next result shows, the comparison is reversed in the case when most consumers are $\alpha$ types.

Proposition 4 There exists a $\overline{\beta}$ such that, if $\beta < \overline{\beta}$, then the optimal equilibrium under no umbrella branding is superior to the optimal equilibrium under umbrella branding.

The intuition for this result is as follows. If the number or $\beta$ buyers is very small, then the equilibrium in Proposition 2 cannot be implemented. Two-product firms must therefore
play separate games with each set of \( \alpha \) buyers. The only effect of umbrella branding is then to allow \( \beta \) buyers to identify the firm as the producer of the two products that such consumers purchase. This turns out to harm the firm, in the following sense: if reputation breaks down in one of the products, then the firm deviates in the other product as well, since the cost of producing one quality product is higher than for a one-product firm. Under umbrella branding, \( \beta \) types know this and immediately stop purchasing the other product, in anticipation of the firm’s reputation breakdown. In other words, umbrella branding implies that bad news travel faster. By contrast, naming the two products with different names allows the firm to avoid this problem.

5 Discussion

Propositions 3 and 4 can be rephrased by stating that umbrella branding is an optimal strategy only when most buyers purchase both of the products sold under the same umbrella. That is, the products must be related to each other by the identity of the consumer. Previous theoretical and empirical studies have emphasized the importance of a good “fit” for the success of a brand extension or umbrella branding strategy. For example, Sappington and Wernerfelt (1985) consider a model of horizontal product differentiation and show that umbrella branding may reduce the consumer’s search risk for a new product if firms add to their umbrella products with identical location as their initial one.\(^8\)

The main difference of my results is that the relation between products is not to be found in the objective characteristics of the products but rather in the set of consumers that buy both products. The Virgin brand, for example, includes products as diverse as CDs, air travel and pension funds. However, if the typical customer of Virgin products (Generation X and late baby boomers) typically buys all of the above products, then it may be said that the products “fit” well with each other.

The idea of consumer identity fit is obviously not novel. For example, Kotler et al. (1996)

\(^8\)Sappington and Wernerfelt (1985) also test, inter alia, the empirical prediction that the probability a new product is umbrella branded is decreasing with respect to the distance from the first product. Data from the U.S. liquor industry is consistent with the prediction.

Aaker and Keller (1990) also find that “attitude toward the extension was higher when . . . there was both a perception of ‘fit’ between the two product classes and a perception of high quality for the original brand.” For example, Crest mouthwash and the Häagen Dazs candy bar were found to be successful brand extensions, whereas McDonald’s photo processing and Heineken popcorn were much less so.
caution that “brand extensions can hurt the core values of the original product” (p. 568). They recall the example of Miller, whose “original beer was advertised to older drinkers on the basis of traditional American values, while Miller Lite targeted the under-24s using tongue-in-cheek endorsements from sportsmen and comics.” My results suggest that, in addition to the subjective elements of a brand’s core values, there is also an objective reason why commonality of customer base is important.

Although the analysis in this paper is couched in terms of quality provision, we can think of its main ideas and results in a broader context. One particular area of application is status goods. In status goods, the customer’s main concern is that the brand be kept exclusive. However, from the seller’s point of view there is a short-run incentive to sell large quantities, though such strategy may lead to brand dilution. This corresponds to a structure that is very similar to the problem of quality provision: (a) mutual gains from increasing quality/status, but (b) moral hazard on the seller’s side, with short-run incentive to cheat consumers. My results then imply that the optimal strategy for branding houses such as LVMH, Gucci or Prada is to stretch (or license) their brand to products that are purchased by the same customer base as their current product line.

6 Concluding remarks

In a recent Marketing textbook (Churchill and Peter, 1998), one reads that

A brand extension strategy can fail and even damage existing products if the brand is used for products that are unrelated or when the brand’s name or image doesn’t fit the new product.

My results agree with this form of conventional wisdom. Specifically, Propositions 3 and 4 suggest that success of an umbrella branding strategy requires that there is a large set of common buyers.

Although I have only considered the possibilities of one versus two products sold under the same name, the results in this paper provide the basis for a theory of optimal umbrella size. Absent cost considerations or other considerations, optimal size would be the maximum

\footnote{Calvin Klein’s current suit against Warnaco, citing the latter for trademark violation and breach of fiduciary duty, is an example of this short-run incentive. See T Agins and R Quick, “Calvin Klein Sues Warnaco, Its Jeans Maker,” The Wall Street Journal, May 31, 2000.}
size such that the equilibrium of Proposition 2 is still viable. A more precise derivation of such optimal size is a promising avenue for future research.
Appendix

**Proof of Proposition 1:** Denote by $V$ the seller’s expected equilibrium payoff, and let $\theta$ be the probability of reputation breakdown when the product fails. If the firm invests in quality until reputation breaks down, expected payoff at the beginning of the game is given by

$$V = \pi - e + \left( (1 - \alpha) + \alpha(1 - \theta) \right) \delta V.$$  \hspace{1cm} (1)

Solving for $V$, we get

$$V = \frac{\pi - e}{1 - \left( (1 - \alpha) + \alpha(1 - \theta) \right) \delta}.$$  \hspace{1cm} (2)

If the seller cheats on quality, then expected payoff is

$$V' = \pi + (1 - \theta)\delta V.$$  \hspace{1cm} (3)

Based on (1) and (3), the equilibrium condition that $V \geq V'$ can be written as

$$\pi - e + \left( (1 - \alpha) + \alpha(1 - \theta) \right) \delta V \geq \pi + (1 - \theta)\delta V,$$

or simply

$$V \geq \frac{e}{1 - \alpha \delta \theta}.$$  \hspace{1cm} (4)

Substituting (2) for $V$ and simplifying, we get

$$\theta \geq \frac{e(1 - \delta)}{\delta (\pi(1 - \alpha) - e)}.$$  \hspace{1cm} (5)

Moreover, the right-hand side of the above inequality is less than one if and only if

$$\alpha \leq 1 - \frac{e}{e \pi}.$$

An optimal equilibrium maximizes discounted payoff $V$ given the no-deviation constraint (4). Since $V$ is decreasing in $\theta$, it is optimal to choose $\theta$ such that (4) is an equality. Part (iv) of the proposition results from taking the derivative with respect to the variables on the right-hand side. ■

**Proof of Proposition 2:** Denote by $V$ the seller’s expected equilibrium payoff, and let $\theta_i$ probability of reputation breakdown when $i$ products fail, $i = 1, 2$. If the firm starts by investing in quality in both products, then expected value at the beginning of the game is given by

$$V = 2\pi - e'' + \left( (1 - \gamma)^2 + 2\gamma(1 - \gamma)\theta_1 + \gamma^2\theta_2 \right) \delta V.$$  \hspace{1cm} (6)

Each term within brackets corresponding to each of three possible events: no product failure, one product failure, and two product failures, respectively. Solving for $V$, we get

$$V = \frac{2\pi - e''}{1 - \left( (1 - \gamma)^2 + 2\gamma(1 - \gamma)\theta_1 + \gamma^2\theta_2 \right) \delta}.$$  \hspace{1cm} (5)
A seller that cheats consumers by lowering the quality of one of its products has an expected payoff

\[ V'_1 = 2\pi - \epsilon' + \left( (1 - \gamma)\theta_1 + \gamma\theta_2 \right) \delta V. \]  

(6)

Finally, a seller that cheats consumers by lowering the quality of both of its products has an expected payoff

\[ V'_2 = 2\pi + \theta_2 \delta V. \]  

(7)

An optimal equilibrium is obtained by

\[
\max_{\theta_1, \theta_2} V \\
\text{s.t.} \quad V \geq V'_1 \\
\quad V \geq V'_2
\]

where \( V, V'_1, V'_2 \) are given by (5)-(7).

Differentiating (5)-(7) with respect to \( V_1, V_2 \), we get

\[
\left. \frac{\partial V_1}{\partial V_2} \right|_{V'_1} = -\frac{\gamma}{2(1 - \gamma)} > \left. \frac{\partial V_1}{\partial V_2} \right|_{V'_2} = -\frac{\gamma}{1 - \gamma} > \left. \frac{\partial V_1}{\partial V_2} \right|_{V'_2} = -\infty,
\]

where \( \left| \cdot \right| \) means “maintaining \( x \) constant.”

Consider an equilibrium such that \( V_1 < V \) and \( V_2 > 0 \). Consider a candidate alternative equilibrium obtained by slightly decreasing \( V_2 \) and increasing \( V_1 \) by \( dV_1 = \frac{\gamma}{1 - \gamma} dV_2 \). This implies that:

(a) the slack of the first constraint (cheating in one product) is maintained; (b) the slack of the second constraint (cheating in both products) is increased; (c) \( V \) increases. (a) and (b) imply that the new equilibrium is indeed an equilibrium; (c) implies that it is better.

Proof of Proposition 4: Suppose that \( \beta \) in infinitesimal. This implies that type \( \beta \) consumers are payoff irrelevant in relation to \( \alpha \) consumers. The optimal solution is therefore as in Proposition 1: if one of the product fails, then type \( \alpha \) consumers stop buying that product with probability \( \theta \). There is however one important difference: if product \( i \)'s reputation breaks down, then a two-product firm no longer has an incentive to maintain its quality effort in the other product. This is a consequence of Assumption 1: the value \( \theta \) is calculated such that a one-product firm is just indifferent between deviating and not deviating; and since \( \epsilon' > \epsilon \), a two-product firm would not have the incentive to maintain the reputation of one product only.

In this context, the only difference between umbrella branding and no-umbrella branding is that type \( \beta \) consumers can identify the firm's second product under umbrella branding. This implies that, as reputation breaks down in one product at time \( t \), type \( \beta \) consumers know that quality will be reduced to zero in the other product beginning at time \( t + 1 \). Under no umbrella branding, however, those consumers would still buy from the firm's second product at time \( t + 1 \). It follows that the equilibrium under no umbrella branding yields a greater payoff.
References


