# Customer-Base Analysis in a Discrete-Time Noncontractual Setting

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**Abstract**

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Many businesses track repeat transactions on a discrete-time basis. These include: (1) companies where transactions can only occur at fixed regular intervals, (2) firms that frequently associate transactions with specific events (e.g., a charity that records whether or not supporters respond to a particular appeal), and (3) organizations that simply use discrete reporting periods even though the transactions can occur at any time. Furthermore, many of these businesses operate in a noncontractual setting, so they have a difficult time differentiating between those customers who have ended their relationship with the firm versus those who are in the midst of a long hiatus between transactions. We develop a model to predict future purchasing patterns for a customer base that can be described by these structural characteristics. Our beta-geometric/beta-Bernoulli (BG/BB) model captures both of the underlying behavioral processes (i.e., customers' purchasing while "alive", and time until each customer permanently "dies"). The model is easy to implement in a standard spreadsheet environment, and yields relatively simple closed-form expressions for the expected number of future transactions conditional on past observed behavior (and other quantities of managerial interest). We apply this discrete-time analog of the well-known Pareto/NBD model to a dataset on donations made by the supporters of a public radio station located in the Midwestern United States. Our analysis demonstrates the excellent ability of the BG/BB model to describe and predict the future behavior of a customer base.

**Keywords:** BG/BB, beta-geometric, beta-binomial, customer-base analysis, customer lifetime value, CLV, RFM, Pareto/NBD

# 1 Introduction

Consider a major public radio station located in the Midwestern United States which, like most public radio stations, is supported in large part by listener contributions. In 1995 the station "acquired" 11,104 first-time supporters; in each of the following six years, these individuals either did or did not support the radio station. As shown in Table 1, donation behavior can be characterized by a binary string, where 1 indicates that a donation was made. (For the purposes of this analysis, we focus only on the incidence on the donations; we ignore the dollar values.) Given these data, management would like to know which individuals are most likely to be active donors in the future, and predict the level of "transactions" they could expect in future years from this cohort of donors (both individually and collectively).

| ID | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 |
|---|---|---|---|---|---|---|---|
| 100001 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 100002 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 100003 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 100004 | 1 | 0 | 1 | 0 | 1 | 1 | 1 |
| 100005 | 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 100006 | 1 | 1 | 1 | 1 | 0 | 1 | 0 |
| 100007 | 1 | 1 | 0 | 1 | 0 | 1 | 0 |
| 100008 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 100009 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| 100010 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| ⋮ | | | ⋮ | | | ⋮ | |
| 111102 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 111103 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 111104 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 1:** Annual donation behavior by the 1995 cohort of first-time supporters.

Management has a five-year planning period, and therefore would like to forecast the expected number of donations for the 1995 cohort as a whole, as well as for particular types of individuals, over the period 2002-2006. For instance:

- What should be expected from donor 100008, who has made a repeat donation in each of the six years since becoming a supporter of the station: is he likely to go "five-for-five" in the future period, or how much "shrinkage" is expected to occur?

- How about comparing donor 100009, who had been a consistent supporter up until 2001, versus donor 100004, who has had a more irregular history, with one less donation overall but with one made in 2001.
- Likewise, how does donor 100004 compare to donor 111103? They've both made four repeat donations including one in 2001, but their earlier histories differ somewhat from each other.
- Finally, how about the many donors (such as 100001) who have done nothing since their initial contributions? Should the station write them off, or is there still some meaningful future value in them — individually and collectively?

Recognizing that this a noncontractual setting,[1] the marketing analyst may think "let's use the Pareto/NBD", a model developed by Schmittlein et al. (1987) to provide answers to the kinds of customer-base analysis questions listed above.

But is this an appropriate way to proceed? At the heart of the Pareto/NBD model is the assumption that customer purchasing while "alive" is characterized by a Poisson distribution and that cross-sectional heterogeneity in the mean purchase rates is characterized by a gamma distribution (resulting in an NBD model of repeat buying while alive). The use of the Poisson distribution assumes that transactions can occur at any point in time; this may be an acceptable assumption for the purchasing of CDs from a web site or for the purchasing of office products in a B2B setting, which are the empirical settings considered by Fader et al. (2005) and Schmittlein and Peterson (1994), respectively. However, it is not a valid assumption in a number of other settings, including the public radio station described above.

As another example, consider attendance at the INFORMS Marketing Science Conference. The conference occurs at a discrete point in time and an individual can either attend, or not.

[1] In a contractual setting (e.g., gym membership, cable TV, theater subscription plan) we observe the time at which the customer "dies" (i.e., ends their relationship with the firm). In a noncontractual setting (e.g., traditional mail order, retail store patronage), however, the time at which a customer dies is unobserved by the firm; customers do not notify the firm "when they stop being a customer. Instead they just silently attrite" (Mason 2003, p. 55). The only potential evidence of this having happened is an unusually long hiatus since the last recorded purchase. The challenge facing the analyst is how to differentiate between those customers who have ended their relationship with the firm versus those who are simply in the midst of a long hiatus between transactions.

Similarly, consider church service attendance. An individual cannot attend a church service at any random time during the week; she can either attend the Sunday morning service, or not. In both cases, the opportunities for a transaction occur at discrete points in time, and there is an upper bound on the number of transactions that can occur in a fixed unit of time; an individual cannot attend the INFORMS Marketing Science Conference more than once a year, or attend the Sunday morning church service more than 52 times a year. In such noncontractual settings, the behavior is "necessarily discrete" and it is clearly incorrect to model the number of transactions using a Poisson distribution. It would be more appropriate to model the number of transactions in a given time period using a Bernoulli process.

In other settings, the behavior of interest can occur in continuous time, but it is "effectively discrete" in the way firms view it. Consider the case of blood donations. A blood collection agency will send quarterly notices to its donor base, requesting that they give blood. While an individual can give blood at any point in time during that quarter, there is still an upper bound in the number of times the agency is willing to accept blood from any donor and can therefore characterize a donor's behavior in terms of whether or not she gave blood in a fixed time interval. Similarly, a charity may send out letters every six months requesting money. While an individual can send in a donation at any point in time, the charity is basically interested in whether or not he responded to a specific request for funds and will therefore characterize donation behavior simply in terms of whether or not the individual responds to a mailing (Piersma and Jonker 2004). A number of mail-order companies also think of their customer behavior in such a manner (e.g., did the customer place an order in response to the quarterly catalog mailing?). In these cases, it is convenient to think of there being a natural upper bound on the number of transactions that can occur in a fixed unit of time (e.g., year) and it is therefore more appropriate to model the number of transactions using a Bernoulli process rather than a Poisson distribution.

Finally, there are cases where the event of interest has no constraints on it at all — it is truly a continuous-time behavior, but it is so rare per unit of time that management will choose to discretize the purchasing data for analysis and reporting purposes. For example, a cruise-ship company may characterize customer behavior in terms of whether or not each customer went on a cruise in 2000, 2001, 2002, etc. (Berger et al. 2003). Once again, purchasing behavior is more conveniently described as a Bernoulli process, rather than as a Poisson process. An example of this in a CPG setting is the work of Chatfield and Goodhardt (1970), who model the purchasing of a product not in terms of the number of purchases made by an individual in a 24-week period (using the NBD model) but rather in terms of the number of weeks in which an individual purchased the product (using the beta-binomial model with $n = 24$).

Figure 1 illustrates this continuum of settings in which it is either correct or simply makes more sense to model individual-level transaction behavior using a Bernoulli process rather than a Poisson distribution. In all of these settings, it is clearly inappropriate to use the Pareto/NBD as the underlying model for a customer-base analysis exercise.

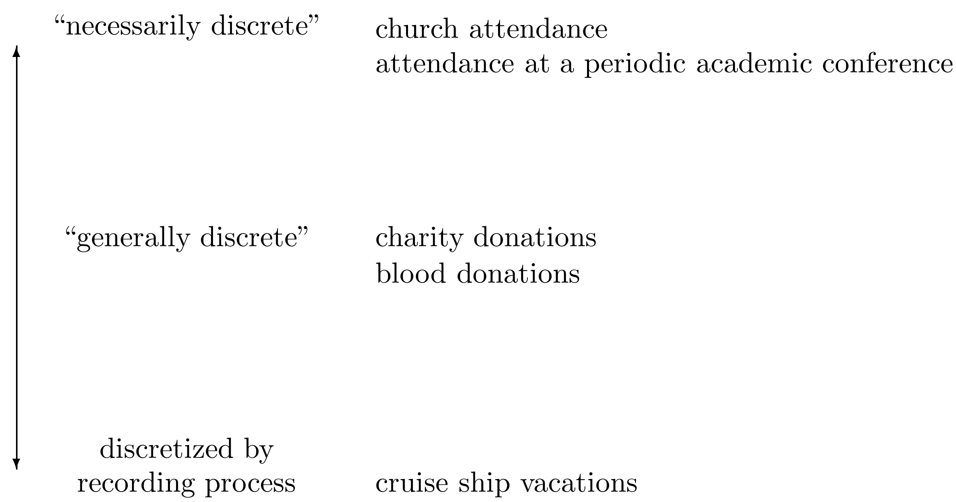



**Figure 1:** Classifying "discrete-time" transaction opportunities.

In this paper we develop a model that can be used to answer the critical customer-base analysis questions in discrete time, noncontractual settings; in other words, we develop a discrete-time analog of the Pareto/NBD model. While many aspects of the Pareto/NBD model (and the inferences frequently associated with it) carry over fairly smoothly to the discrete-time setting, there are a number of interesting issues that arise in the discrete-time setting that are quite unique — and offer significant benefits for model implementation. In the next section, we first outline the assumptions underpinning this model and then present expressions for a number of managerially relevant quantities. This is followed by an empirical analysis (for the aforementioned public radio station) in which we carefully examine the performance of the model both in a six-year calibration sample and a five-year holdout period. We conclude with a discussion of several additional issues that arise from this work.

## 2 Model Development

Our objective is to develop a stochastic model of buyer behavior for discrete-time, noncontractual settings. To start, we define a *transaction opportunity* as either

- a well-defined *point in time* at which a transaction either occurs or does not occur, or
- a well-defined *time interval* during which a (single) transaction either occurs or does not occur.

The first type of transaction opportunity corresponds to the "necessarily discrete" case in Figure 1. The second type of transaction opportunity corresponds to the "generally discrete" and "discretized by the recording process" cases in Figure 1. In all three cases, a customer's transaction history can be expressed as a binary string, where $y_t = 1$ if a transaction occurred at/during the $t$th transaction opportunity, 0 otherwise (for $t = 1, \ldots, n$ transaction opportunities). Note that we are simply interested in modeling the transaction process (i.e., the pattern of 1s and 0s). We are not interested in modeling other behaviors associated with each transaction (e.g., the quantity purchased); this is discussed in Section 4.

Our model is based on the following six assumptions:

i. A customer's relationship with the firm has two phases: he is "alive" (A) for some period of time, then becomes permanently inactive ("dies", D).

ii. While alive, the customer buys at any given transaction opportunity with probability $p$:

$$P(Y_t = 1 \mid p, \text{alive at } t) = p\,.$$

(This implies that the number of transactions by a customer alive for $n$ transaction opportunities follows a binomial distribution.)

iii. A "living" customer "dies" at the beginning of a transaction opportunity with probability $\theta$. (This implies that the (unobserved) lifetime of a customer is characterized by a geometric distribution.)

iv. Heterogeneity in $p$ follows a beta distribution with pdf

$$f(p \mid \alpha, \beta) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < p < 1\,. \tag{1}$$

v. Heterogeneity in $\theta$ follows a beta distribution with pdf

$$f(\theta \mid \gamma, \delta) = \frac{\theta^{\gamma-1}(1-\theta)^{\delta-1}}{B(\gamma, \delta)}, \quad 0 < \theta < 1\,. \tag{2}$$

vi. The transaction probability $p$ and the dropout probability $\theta$ vary independently across customers.

Assumptions (ii) and (iv) yield the beta-Bernoulli model (i.e., the beta-binomial model without the binomial coefficient). Similarly, assumptions (iii) and (v) yield the beta-geometric (BG) distribution. We therefore call this the beta-geometric/beta-Bernoulli (BG/BB) model of buyer behavior.

## 2.1 Derivation of Model Likelihood Function

Consider a customer with repeat purchase string 1 0 1 0 0. What is $P(Y_1 = 1, Y_2 = 0, Y_3 = 1, Y_4 = 0, Y_5 = 0 \mid p, \theta)$? The fact that the customer made a purchase at the third transaction opportunity means that he must have been alive for t = 1, 2, 3. However, $Y_4 = 0, Y_5 = 0$ could be the result of one of three scenarios: i) he died at the beginning of the fourth transaction opportunity ($AAADD$), ii) he was alive at the fourth transaction opportunity and died at the beginning of the fifth transaction opportunity ($AAAAD$), and iii) he was alive at both the fourth and fifth transaction opportunities ($AAAAA$). We therefore compute $P(Y_1 = 1, Y_2 = 0, Y_3 = 1, Y_4 = 0, Y_5 = 0 \mid p, \theta)$ by computing the probability of the purchase string conditional on each scenario and multiply it by the probability of that scenario:

$$
\begin{aligned}
f(10100\,|\,p,\theta) &= f(10100\,|\,p,\text{AAADD})P(\text{AAADD}\,|\,\theta) \\
&\qquad + f(10100\,|\,p,\text{AAAAD})P(\text{AAAAD}\,|\,\theta) \\
&\qquad\qquad + f(10100\,|\,p,\text{AAAAA})P(\text{AAAAA}\,|\,\theta) \\
&= p(1-p)p\,\underbrace{(1-\theta)^3\theta}_{P(\text{AAADD})} + p(1-p)p(1-p)\,\underbrace{(1-\theta)^4\theta}_{P(\text{AAAAD})} \\
&\qquad + \underbrace{p(1-p)p}_{P(Y_1=1,Y_2=0,Y_3=1)}\,(1-p)(1-p)\,\underbrace{(1-\theta)^5}_{P(\text{AAAAA})}
\end{aligned} \tag{3}
$$

Note that the zero-order nature of purchasing while the customer is alive means that the exact order of any given number of transactions prior to the last observed transaction does not matter. For example, it should be clear that $f(10100\,|\,p,\theta) = f(01100\,|\,p,\theta)$. Therefore we do not need the complete binary-string representation of a customer's transaction history. Rather, all we need to know for $n$ transaction opportunities are *frequency* and *recency*: the number of transactions across the calibration period ($x = \sum_{t=1}^{n} y_t$), and the transaction opportunity at which the last observed transaction occurred ($t_x$).[2] We therefore go from $2^n$ binary string representations of all the possible purchase patterns to $n(n+1)/2+1$ possible recency/frequency patterns.

This realization that recency and frequency are sufficient summary statistics offers signficant benefits for model implementation, particularly as the number of transaction opportunities becomes sizeable. For instance, in the case of our public radio station, we can compress the number of necessary binary strings from 64 down to 22 recency/frequency combinations, making it a bit easier to visualize and manipulate the dataset. But in another recent application with $n = 10$, we saw a reduction from 1024 binary strings down to 56 recency/frequency combinations. Furthermore, these numbers are not affected by the size of the customer base being modeled; see Table 2 for a complete characterization of the public radio dataset partially presented in Table 1. Whether we have 11,000 customers or 11 million customers, the data structure is effectively identical — the numbers in the # donors columns would grow, but the computational demands for data storage and manipulation are unchanged.

[2] If $x = 0$, then $t_x = 0$. Note that this measure of recency differs from that normally used by the direct marketing community, who measure recency as the time from the last observed transaction to the end of the observation period (i.e., $n - t_x$).

| $x$ | $t_x$ | # donors | $x$ | $t_x$ | # donors |
|---|---|---|---|---|---|
| 6 | 6 | 1203 | 4 | 4 | 240 |
| 5 | 6 | 728 | 3 | 4 | 181 |
| 4 | 6 | 512 | 2 | 4 | 155 |
| 3 | 6 | 357 | 1 | 4 | 78 |
| 2 | 6 | 234 | 3 | 3 | 322 |
| 1 | 6 | 129 | 2 | 3 | 255 |
| 5 | 5 | 335 | 1 | 3 | 129 |
| 4 | 5 | 284 | 2 | 2 | 613 |
| 3 | 5 | 225 | 1 | 2 | 277 |
| 2 | 5 | 173 | 1 | 1 | 1091 |
| 1 | 5 | 119 | 0 | 0 | 3464 |

**Table 2:** Recency/frequency summary of the annual donation behavior by the 1995 cohort of first-time supporters ($n = 6$).

Returning to the likelihood function, we generalize the logic behind the construction of (3), so it follows that

$$L(p, \theta \,|\, x, t_x, n) = p^x(1-p)^{n-x}(1-\theta)^n + \sum_{i=0}^{n-t_x-1} p^x(1-p)^{t_x-x+i}\theta(1-\theta)^{t_x+i} \tag{4}$$

To arrive at the likelihood function for a randomly chosen customer with purchase history $(x, t_x, n)$, we remove the conditioning on $p$ and $\theta$ by taking the expectation of (4) over their respective mixing distributions:

$$\begin{aligned} L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n) &= \int_0^1 \int_0^1 L(p, \theta \,|\, x, t_x, n) f(p \,|\, \alpha, \beta) f(\theta \,|\, \gamma, \delta)\, dp\, d\theta \\ &= \frac{B(\alpha+x, \beta+n-x)}{B(\alpha, \beta)} \frac{B(\gamma, \delta+n)}{B(\gamma, \delta)} \\ &\quad + \sum_{i=0}^{n-t_x-1} \frac{B(\alpha+x, \beta+t_x-x+i)}{B(\alpha, \beta)} \frac{B(\gamma+1, \delta+t_x+i)}{B(\gamma, \delta)}\,. \end{aligned} \tag{5}$$

(The solution to the double integral follows naturally from the integral representation of the beta function.)

The four BG/BB model parameters $(\alpha, \beta, \gamma, \delta)$ can be estimated via the method of maximum likelihood in the following manner. For a calibration period with $n$ transaction opportunities, we have $I = n(n+1)/2 + 1$ possible recency/frequency patterns, each containing $f_i$ customers.

The sample log-likelihood function is given by

$$LL(\alpha,\beta,\gamma,\delta)=\sum_{i=1}^{I} f_i \ln\left[L(\alpha,\beta,\gamma,\delta\,|\,x_i,t_{x_i},n)\right]$$

where $x_i$ and $t_{x_i}$ are the frequency and recency for each unique pattern. This can be maximized using standard numerical optimization routines. These calculations are easy to perform in a spreadsheet environment; in fact, the entire model implementation (from initial data setup through the calculation of the "key results" in the next section) rarely requires the analyst to use any software beyond a spreadsheet. This is a major benefit of the BG/BB model.

## 2.2 Key Results

We now present expressions for a set of quantities of interest to anyone wanting to apply this model of buyer behavior in a discrete-time, noncontractual setting. (The associated derivations can be found in the appendix.)

Let the random variable $X(n)=\sum_{t=1}^{n} Y_t$ denote the number of transactions occurring across the first $n$ transaction opportunities. The BG/BB pmf is

$$\begin{aligned}P(X(n)=x\,|\,\alpha,\beta,\gamma,\delta)=&\binom{n}{x}\frac{B(\alpha+x,\beta+n-x)}{B(\alpha,\beta)}\frac{B(\gamma,\delta+n)}{B(\gamma,\delta)}\\&+\sum_{i=x}^{n-1}\binom{i}{x}\frac{B(\alpha+x,\beta+i-x)}{B(\alpha,\beta)}\frac{B(\gamma+1,\delta+i)}{B(\gamma,\delta)}\,,\end{aligned}\tag{6}$$

with mean

$$E(X(n)\,|\,\alpha,\beta,\gamma,\delta)=\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{1}{\gamma-1}\right)\left\{\delta-\frac{\Gamma(\gamma+\delta)\Gamma(\delta+n+1)}{\Gamma(\delta)\Gamma(\gamma+\delta+n)}\right\}.\tag{7}$$

More generally, let the random variable $X(n,n+n^*)=\sum_{t=n+1}^{n*} Y_t$ denote the number of transactions in the interval $(n,n+n^*]$. The BG/BB probability of $x^*$ transactions occurring in this interval is given by

$$
\begin{aligned}
P(X(n, n+n^*) = x^* \mid \alpha, \beta, \gamma, \delta) = \delta_{x^*=0} &\left\{ 1 - \frac{B(\gamma, \delta+n)}{B(\gamma, \delta)} \right\} \\
&+ \binom{n^*}{x^*} \frac{B(\alpha+x^*, \beta+n^*-x^*)}{B(\alpha, \beta)} \frac{B(\gamma, \delta+n+n^*)}{B(\gamma, \delta)} \\
&+ \sum_{i=x^*}^{n^*-1} \binom{i}{x^*} \frac{B(\alpha+x^*, \beta+i-x^*)}{B(\alpha, \beta)} \frac{B(\gamma+1, \delta+n+i)}{B(\gamma, \delta)} ,
\end{aligned} \tag{8}
$$

with mean

$$
\begin{aligned}
E(X(n, n+n^*) \mid \alpha, \beta, \gamma, \delta) = &\left( \frac{\alpha}{\alpha+\beta} \right) \left( \frac{1}{\gamma-1} \right) \\
&\times \left\{ \frac{\Gamma(\gamma+\delta)\Gamma(\delta+n+1)}{\Gamma(\delta)\Gamma(\gamma+\delta+n)} - \frac{\Gamma(\gamma+\delta)\Gamma(\delta+n+n^*+1)}{\Gamma(\delta)\Gamma(\gamma+\delta+n+n^*)} \right\} .
\end{aligned} \tag{9}
$$

In most customer-base analysis settings we are interested in making statements about customers conditional on their observed purchase history $(x, t_x, n)$.

- The probability that a customer with purchase history $(x, t_x, n)$ will be alive at the $(n+1)th$ transaction opportunity is

$$
\begin{aligned}
&P(\text{alive at } n+1 \mid \alpha, \beta, \gamma, \delta, x, t_x, n) \\
&\quad = \frac{B(\alpha+x, \beta+n-x)}{B(\alpha, \beta)} \frac{B(\gamma, \delta+n+1)}{B(\gamma, \delta)} \Big/ L(\alpha, \beta, \gamma, \delta \mid x, t_x, n) .
\end{aligned} \tag{10}
$$

- The probability that a customer with purchase history $(x, t_x, n)$ makes $x^*$ transactions in the interval $(n, n+n^*]$ is

$$
\begin{aligned}
&P(X(n, n+n^*) = x^* \mid \alpha, \beta, \gamma, \delta, x, t_x, n) \\
&\quad = \delta_{x^*=0} \left\{ 1 - \frac{\mathcal{C}_1}{L(\alpha, \beta, \gamma, \delta \mid x, t_x, n)} \right\} + \frac{\mathcal{C}_2}{L(\alpha, \beta, \gamma, \delta \mid x, t_x, n)}
\end{aligned} \tag{11}
$$

where

$$
\mathcal{C}_1 = \frac{B(\alpha+x, \beta+n-x)}{B(\alpha, \beta)} \frac{B(\gamma, \delta+n)}{B(\gamma, \delta)}
$$

and

$$\mathcal{C}_2 = \binom{n^*}{x^*}\frac{B(\alpha+x+x^*,\beta+n-x+n^*-x^*)}{B(\alpha,\beta)}\frac{B(\gamma,\delta+n+n^*)}{B(\gamma,\delta)} + \sum_{i=x^*}^{n^*-1}\binom{i}{x^*}\frac{B(\alpha+x+x^*,\beta+n-x+i-x^*)}{B(\alpha,\beta)}\frac{B(\gamma+1,\delta+n+i)}{B(\gamma,\delta)}\,.$$

- The expected number of future transactions across the next $n^*$ transaction opportunities by a customer with purchase history $(x, t_x, n)$ is

$$E(X(n,n+n^*)\,|\,\alpha,\beta,\gamma,\delta,x,t_x,n) = \frac{1}{L(\alpha,\beta,\gamma,\delta\,|\,x,t_x,n)}\frac{B(\alpha+x+1,\beta+n-x)}{B(\alpha,\beta)} \times \frac{\Gamma(\gamma+\delta)}{(\gamma-1)\Gamma(\delta)}\left\{\frac{\Gamma(\delta+n+1)}{\Gamma(\gamma+\delta+n)} - \frac{\Gamma(\delta+n+n^*+1)}{\Gamma(\gamma+\delta+n+n^*)}\right\}. \tag{12}$$

- We may also be interested in making inferences about a customer's latent transaction and dropout probabilities. The mean of the marginal posterior distribution of $p$ is

$$E(P\,|\,\alpha,\beta,\gamma,\delta,x,t_x,n) = \left(\frac{\alpha}{\alpha+\beta}\right)\frac{L(\alpha+1,\beta,\gamma,\delta\,|\,x,t_x,n)}{L(\alpha,\beta,\gamma,\delta\,|\,x,t_x,n)}\,, \tag{13}$$

while the mean of the marginal posterior distribution of $\theta$ is

$$E(\Theta\,|\,\alpha,\beta,\gamma,\delta,x,t_x,n) = \left(\frac{\gamma}{\gamma+\delta}\right)\frac{L(\alpha,\beta,\gamma+1,\delta\,|\,x,t_x,n)}{L(\alpha,\beta,\gamma,\delta\,|\,x,t_x,n)}\,. \tag{14}$$

- Many customer-base analysis exercises are motivated by a desire to compute customer lifetime value (CLV), which is "the present value of future cash flows attributed to the customer relationship" (Pfeifer, Haskins, and Conroy 2005, p. 10). The general explicit formula for computing CLV is (Rosset et al. 2003)

$$E(CLV) = \int_0^\infty E[v(t)]S(t)d(t)dt\,,$$

where $E[v(t)]$ is the expected value of the customer at time $t$ (assuming he is alive), $S(t)$ is the survivor function, and $d(t)$ is a discount factor that reflects the present value of money received at time $t$. Following Fader et al. (2005), if we assume that the process describing the net cash flow per transaction for a given customer is both independent of the transaction process and stationary, we can express $v(t)$ as net cash flow / transaction$\times t(t)$, where $t(t)$ is the transaction rate at $t$.

In many cases we are interested in the expected *residual* lifetime of a customer. Standing at time $T$,

$$E(RLV) = E(\text{net cashflow} \,/\, \text{transaction}) \times \underbrace{\int_T^\infty E[t(t)]S(t \mid t > T)d(t-T)dt}_{\text{discounted expected residual transactions}}.$$

The number of discounted expected residual transactions ($DERT$) is the present value of the expected future transaction stream for a customer with purchase history $(x, t_x, T)$. Fader et al. (2005) derive the expression for this quantity when the transaction process can be described by the Pareto/NBD model. When the transaction process is described by the BG/BB model, the present value of the expected number of future transactions for a customer with purchase history $(x, t_x, n)$, with discount rate $d$ is:[3]

$$\begin{aligned} DERT(d \mid \alpha, \beta, \gamma, \delta, x, t_x, n) = {} & \frac{B(\alpha + x + 1, \beta + n - x)}{B(\alpha, \beta)} \frac{B(\gamma, \delta + n + 1)}{B(\gamma, \delta)(1+d)} \\ & \times \frac{{}_2F_1\big(1, \delta + n + 1; \gamma + \delta + n + 1; \frac{1}{1+d}\big)}{L(\alpha, \beta, \gamma, \delta \mid x, t_x, n)}. \end{aligned} \tag{15}$$

where ${}_2F_1(\cdot)$ is the Gaussian hypergeometric function. This number of discounted expected transactions can then be rescaled by the customer's value multiplier to yield an overall estimate of $E(RLV)$. While the presence of the Gaussian hypergeometric function makes this calculation a bit more complex than the others in this section, it is worth emphasizing that it only needs to be evaluated once for any given value of $n$ (i.e., only once per cohort, not for every recency/frequency pattern) and it is relatively straightforward to use a recursion formula to perform the calculations in a familiar spreadsheet environment. Furthermore, this calculation for DERT is far simpler than the equivalent expression derived by Fader et al. (2005) for the Pareto/NBD model. In that case, the DERT expression

[3] Suppose there are $k$ transaction opportunities per year. An annual discount rate of $r$ maps to a discount rate of $d = (1 + r)^{1/k} - 1$.

required the evaluation of the confluent hypergeometric function of the second kind, which is more unfamiliar and burdensome (from a computational standpoint) than the Gaussian hypergeometric function.

# 3 Empirical Analysis

We examine the performance of the BG/BB model using data on annual donation behavior by the supporters of a public radio station located in the Midwestern United States. The full dataset contains information on the 56,847 people who made their first-ever annual donation between 1995 and 2000 (inclusive), from their first year up to and including 2006; the sizes of each annual cohort are given in Table 3.

| Cohort | Size |
|---|---|
| 1995 | 11,104 |
| 1996 | 10,057 |
| 1997 | 9,043 |
| 1998 | 8,175 |
| 1999 | 8,977 |
| 2000 | 9,491 |

**Table 3:** Number of new supporters each year (1995 – 2000).

Our initial analysis focuses on the 11,104 members of the 1995 cohort. We fit the model using the data on whether or not these supporters made repeat donations across 1996 – 2001, and examine the model's predictive performance across a 2002 – 2006 holdout validation period. We follow up this analysis with one in which we pool the six cohorts, fitting the model to the repeat donation data up to and including 2001 and examining its predictive performance over 2002 – 2006. (For the sake of linguistic simplicity, we will refer to the act of making a repeat donation in any given year as making a repeat transaction or purchase.)

## 3.1 Analysis of the 1995 Cohort

The group of 11,104 people that became supporters of the radio station for the first time in 1995 made a total of 24,615 repeat transactions over the next 6 years. The maximum likelihood estimates of the model parameters are reported in Table 4.[4] (We also report the model parameters and value of the log-likelihood function for the beta-Bernoulli model, and note that the addition of the "death" component results in a major improvement in model fit.)

| | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | LL |
|---|---|---|---|---|---|
| BB | 0.487 | 0.826 | | | −35,516.1 |
| BG/BB | 1.204 | 0.750 | 0.657 | 2.783 | −33,225.6 |

**Table 4:** Parameter estimates, 1995 cohort.

The expected number of people making $0, 1, \ldots, 6$ repeat transactions between 1996 and 2001 is computed using (6) and compared to the actual frequency distribution in Figure 2. We note that the model provides a very good fit to the data.

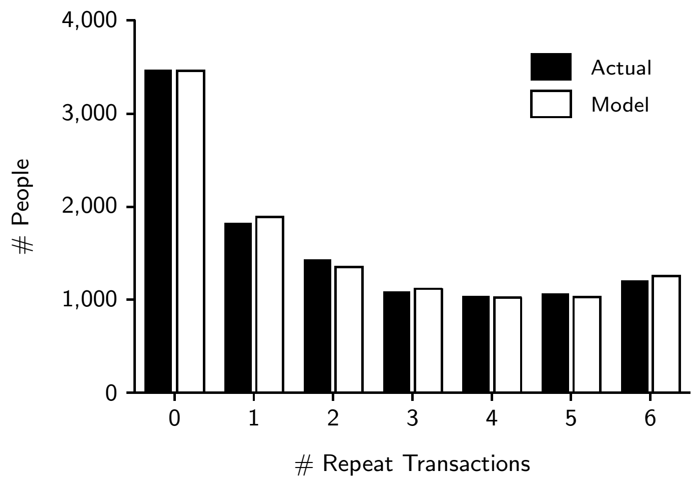


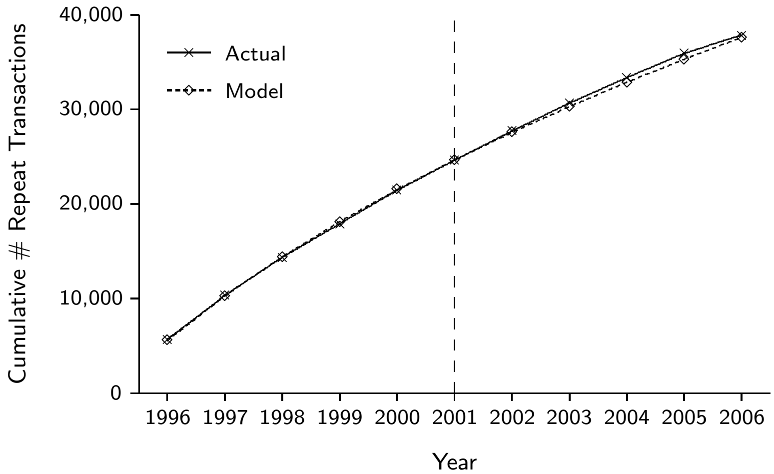

**Figure 2:** Predicted versus actual frequency of repeat transactions.

The performance of the model becomes more impressive when we see how well it tracks repeat transactions over time. Using the expression for the expected number of transactions across $n$ transaction opportunities as given in (7), we compute the expected number of repeat transactions made by the whole cohort of 11,104 people up to 2006. These are plotted along with the actual cumulative numbers in Figure 3a. We note that the BG/BB model predictions accurately track the actual cumulative number of repeat transactions in both the six-year calibration period and the five-year forecast period, underforecasting at 2006 by a mere −0.65%.[5] Further insight

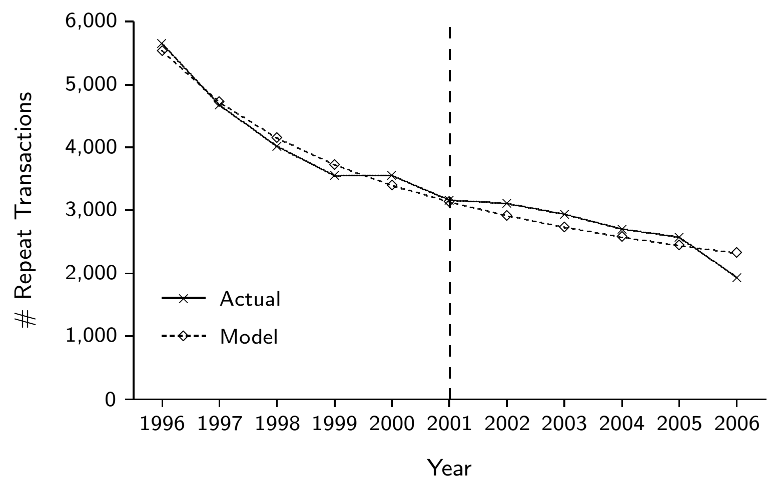
[4] Note that the entire analysis can easily be performed in a spreadsheet environment; copies of the Excel spreadsheets used to perform the analysis presented in this paper are available from the authors.

[5] As a point of comparison, the prediction associated with the BB model overforecasts cumulative repeat transactions at the end of 2006 by 20%.

into the excellent tracking performance of the model is given in Figure 3b, which reports these numbers on a year-by-year basis; we note that the BG/BB model clearly captures the underlying trend in repeat transactions over this fairly lengthy period of time.




(a)




(b)

**Figure 3:** Predicted versus actual (a) cumulative and (b) annual repeat transactions.

To get a clearer idea of how well the model captures validation period purchasing, we compute the expected number of people making $x^* = 0, 1, \ldots, 5$ transactions in 2002–2006 ($n^* = 5$) using (8) and compare it to the actual frequency distribution in Figure 4. We note that the model provides a very good prediction of the actual behavior.

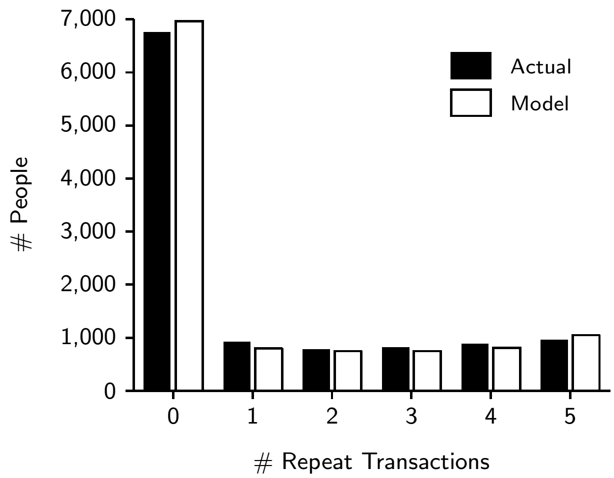



**Figure 4:** Predicted versus actual frequency of repeat transactions in 2002 – 2006.

**Conditional Expectations**

Perhaps a more important examination of the predictive performance of the model focuses on the quality of the predictions of future behavior conditional on past behavior. We use (12) to compute the expected number of transactions in the 2002 – 2006 period ($n^* = 5$) conditional on each of the 22 $(x, t_x)$ patterns associated with $n = 6$. These conditional expectation are reported in Table 5 as a function of recency (the year of the individual's last transaction) and frequency (the number of repeat transactions).

| # Rpt Transactions | Year of Last Transaction | | | | | | |
|---|---|---|---|---|---|---|---|
| (1996 – 2001) | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 |
| 0 | 0.07 | | | | | | |
| 1 | | 0.09 | 0.31 | 0.59 | 0.84 | 1.02 | 1.15 |
| 2 | | | 0.12 | 0.54 | 1.06 | 1.44 | 1.67 |
| 3 | | | | 0.22 | 1.03 | 1.80 | 2.19 |
| 4 | | | | | 0.58 | 2.03 | 2.71 |
| 5 | | | | | | 1.81 | 3.23 |
| 6 | | | | | | | 3.75 |

**Table 5:** Expected number of repeat transactions in 2002 – 2006 as a function of recency and frequency.

In Figure 5a we report these conditional expectations along with the average of the number of the transactions that actually occurred in the 2002 – 2006 forecast period, broken down by the number of repeat transactions in 1996 – 2001. (For each $x$, we are averaging over customers with different values of $t_x$.) Similarly, Figure 5b reports these conditional expectations along with the average of the number of the transactions that actually occurred in the 2002 − 2006 forecast period, broken down by the year of the individual's last transaction. (For each $t_x$, we are averaging over customers with different values of $x$.) We observe that the BG/BB model generates very good predictions of the expected behavior in the longitudinal holdout period, with the only real blemish being an underestimation of expected purchasing by those individuals whose last repeat purchase occurred before 1998.

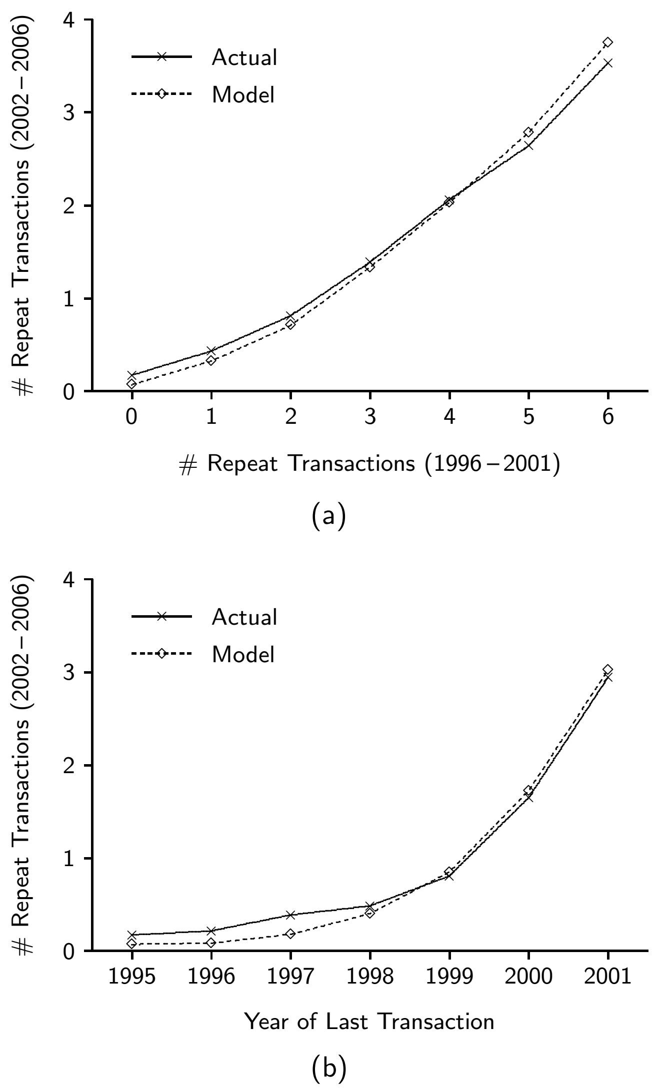



**Figure 5:** Predicted versus actual conditional expectations of repeat transactions in 2002 − 2006 as a function of (a) frequency and (b) recency.

Referring back to Table 5, we can now address the questions about different kinds of customers raised at the outset of the paper.

- A donor who has made repeat transactions at every possible opportunity is expected to make "only" 3.75 transactions over the next five years. Of course, such donors are still extremely valuable, but the possibility of "death" plus the fact that they might have been somewhat lucky in the past make them a bit less valuable than they might have otherwise seemed. (With reference to Figure 5a, we see that this conditional expectation overestimates the actual mean (3.53) by only 6%.)

- Donor 100009, who had a perfect record until the most recent year, is expected to make 1.81 transactions over the next five years. In contrast, donor 100004, with better recency but lower frequency, is expected to make 2.71 transactions over the same period — an increase of nearly 50%! This highlights the critically important role of recency, which can be also seen in the steep growth of the curve in Figure 5b.

- Although donors 100004 and 111103 have different histories, their recency and frequency numbers are identical ($x = 4$, $t_x = 6$); thus, they have the same conditional expectation. Minor, remote differences in purchase histories are deemed to be irrelevant when making predictions using the BG/BB model.

- A donor who has been completely absent since making their initial transaction is expected to make only 0.07 repeat transactions over the next five years. But while each such donor is not particularly valuable alone, it is important to note, as per Table 2, that over 30% of the entire cohort of donors is in this recency/frequency group. Taken together, these donors are expected to make over 240 transactions over the next five years, making them collectively more valuable than about half of the other recency/frequency groups.

Beyond these specific analyses, Table 5 offers additional insights about the broader interplay between recency and frequency. First, note that for any row (i.e., value of $x$), the expected number of transactions in the forecast period decreases as we move from right to left (i.e., the less recent the last observed transaction). This is as we would expect, since the longer the hiatus in making a purchase, the more likely it is that the customer is "dead". Looking down the columns, however, we see a somewhat different pattern. We first look at 2001 and note that the conditional expectation is clearly an increasing function of the number of repeat transactions made in the six-year calibration period. Looking at the 1997−2000 columns, though, we note that the numbers first increase then decrease as the number of repeat transactions made in the six-year calibration period decreases. (A similar pattern is observed in the DERT numbers under the Pareto/NBD model reported in Fader et al. (2005).)

To help understand why this is the case, we use (10) and (13) to compute $P$(alive in 2002) and the mean of the marginal posterior distribution of $p$ as a function of recency and frequency. The combinations of the patterns we shall see in these two tables provides an explanation for this somewhat surprising pattern of conditional expectations.

Let us first consider the mean of the marginal posterior distribution of $p$, Table 6. Looking at this table column-by-column, we see that the posterior mean decreases as a function of the number of repeat transactions in the calibration period for any given value of recency. This is intuitive: the lower the number of repeat transactions reflects a lower underlying probability of purchasing at any given transaction opportunity (assuming one is alive). Perhaps less immediately intuitive is the within-row pattern: for a given level of frequency, the underlying probability of purchasing at any given transaction opportunity increases as recency decreases. The reason for this is that, other things being equal, the longer the hiatus since the last transaction, the more likely it is that the customer is dead and therefore the individual must have had a higher $p$ in order to have the realized number of transactions while alive.

| # Rpt Transactions | Year of Last Transaction | | | | | | |
|---|---|---|---|---|---|---|---|
| (1996−2001) | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 |
| 0 | 0.49 | | | | | | |
| 1 | | 0.66 | 0.44 | 0.34 | 0.30 | 0.28 | 0.28 |
| 2 | | | 0.75 | 0.54 | 0.44 | 0.41 | 0.40 |
| 3 | | | | 0.80 | 0.61 | 0.54 | 0.53 |
| 4 | | | | | 0.82 | 0.68 | 0.65 |
| 5 | | | | | | 0.83 | 0.78 |
| 6 | | | | | | | 0.91 |

**Table 6:** Posterior mean of $p$ as a function of recency and frequency.

Table 7 reports the probability that a customer is alive in 2002 as a function of recency and frequency. Looking across the columns for any value of $x$, the observed pattern is as would be expected, with a lower probability of being alive the longer the hiatus in making a donation.

Taking a column-wise view, the first thing to note is that all customers who made a transaction in 2001 have the same probability of being alive the following year, regardless of the number of repeat transactions they had prior to that year; this is a natural consequence of the Bernoulli "death" process. Looking at the 1997 − 2000 columns, we note that the numbers increase as the number of repeat transactions made in the six-year calibration period decreases. The logic behind this is as follows: looking at the 2000 column, those customers who made only one repeat transaction will have a lower value of $p$ than those who have made a repeat purchase in all five years, and therefore the fact that no transaction occurred in 2001 can be attributed more to their low probability of making a purchase in any given year than to the possibility of them being dead.

| # Rpt Transactions | Year of Last Transaction | | | | | | |
|---|---|---|---|---|---|---|---|
| (1996 − 2001) | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 |
| 0 | 0.11 | | | | | | |
| 1 | | 0.07 | 0.25 | 0.48 | 0.68 | 0.83 | 0.93 |
| 2 | | | 0.07 | 0.30 | 0.59 | 0.80 | 0.93 |
| 3 | | | | 0.10 | 0.44 | 0.77 | 0.93 |
| 4 | | | | | 0.20 | 0.70 | 0.93 |
| 5 | | | | | | 0.52 | 0.93 |
| 6 | | | | | | | 0.93 |

**Table 7:** $P$(alive in 2002) as a function of recency and frequency.

Further insights can be obtained by looking at the marginal posterior distributions of $p$ and $\theta$. With reference to Figure 6a, the prior is the plot of a beta distribution with parameters $\alpha = 1.204$ and $\beta = 0.750$; the overall mean of $p$ across the whole sample is 0.62; with reference to Figure 6b, the prior is the plot of a beta distribution with parameters $\gamma = 0.657$ and $\delta = 2.783$; the overall mean of $\theta$ across the whole sample is 0.19. The distribution of $p$ for an individual who made three consecutive repeat purchases with the last one in 1998 has most of its mass to the right; the observed sequence of purchases reflects the high mean of this distribution ($\bar{p} = 0.80$). At the same time, the three-year hiatus suggests that the supporter is dead as a result of their $\theta$ coming from a posterior distribution with an interior mode and a mean of $\bar{\theta} = 0.20$.

On the other hand, someone who made three repeat purchases with the last one in 2001 had to be alive over the whole period, which is a result of their $\theta$ coming from a beta distribution

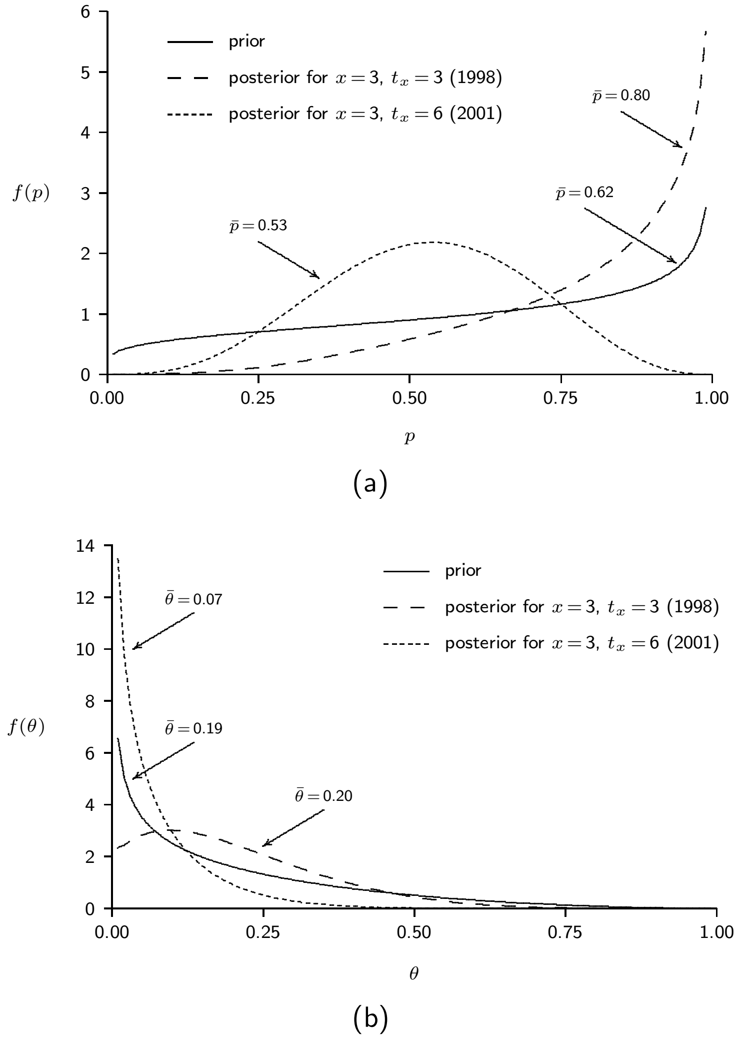



**Figure 6:** Prior and selected posterior distributions of (a) $p$ and (b) $\theta$.

with most of its mass piled to the left and a mean of $\bar{\theta} = 0.07$. The fact that transactions did not occur in three of the six years reflects the fact that their $p$ comes from a distribution with a lower mean ($\bar{p} = 0.53$).

### Conditional Penetration

Ever since the publication of Schmittlein et al. (1987), researchers have shown interest in the $P$(alive) measure. While we have reported this quantity as a means of understanding patterns of conditional expectations, we feel that the measure is of little use by itself. It is a prediction of something that is, by definition, unobservable — whether or not a customer is still alive at a particular point in time. What we feel is more useful is a prediction of whether or not the customer will be *active* in the future, that is, whether or not the customer undertakes any transactions in a specified future period of time.[6]

The probability that a customer is active in the 2002 – 2006 period ($n^* = 5$) is computed as $1 - P(X(n, n + n^*) = 0 \,|\, x, t_x, n)$ using (11), conditional on each of the 22 ($x, t_x$) patterns associated with $n = 6$. This conditional penetration is reported in Table 8 as a function of recency (the year of the individual's last transaction) and frequency (the number of repeat transactions).

| # Rpt Transactions | Year of Last Transaction | | | | | | |
|---|---|---|---|---|---|---|---|
| (1996 – 2001) | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 |
| 0 | 0.05 | | | | | | |
| 1 | | 0.05 | 0.17 | 0.32 | 0.46 | 0.56 | 0.62 |
| 2 | | | 0.05 | 0.24 | 0.48 | 0.66 | 0.76 |
| 3 | | | | 0.09 | 0.40 | 0.69 | 0.84 |
| 4 | | | | | 0.19 | 0.66 | 0.88 |
| 5 | | | | | | 0.51 | 0.91 |
| 6 | | | | | | | 0.92 |

**Table 8:** Probability of being active in 2002 – 2006 as a function of recency and frequency.

Comparing Tables 5 and 8, we note that the estimated probabilities of being alive in 2002 are strictly higher than the corresponding conditional 2002 – 2006 penetration numbers. This makes intuitive sense, but the differences between these measures reflect several factors. First, the $P$(alive) numbers are just for one year, whereas the penetration numbers are for a five-year period. Second, the mere fact that someone is alive does not mean she will be active, as the latter state depends on the person's underlying transaction probability $p$. This is very clear when we look at the right-most column of both tables. While those people who made a purchase in 2001 have the same probability of being alive, irrespective of frequency, their corresponding probabilities of making at least one transaction in the next five years clearly (and logically) decrease as a function of frequency, reflecting in part the lower probabilities of making a purchase at any given transaction opportunity given alive (Table 6). Third, the lower penetration numbers also reflect the fact that inactivity may be due to the person dying in

[6] Many authors, including Schmittlein et al. (1987), have used the terms "alive" and "active" as synonyms. We feel that this should not be the case, with the term "alive" referring to an unobservable state and the term "active" referring to observable behavior.

2003 – 2006, even if they were alive in 2002.

In summary, we encourage researchers who might be attracted by the $P(\text{alive})$ measure to focus on the conditional penetration numbers instead, since they reflect an observable quantity (i.e., whether or not the customer is active).

## 3.2 Pooled Analysis

The analyses presented above all focused on a single cohort, the group of individuals who made their first-ever donation during 1995. However, as noted earlier, we have data for a total of six cohorts. At first glance we may be tempted to apply the model cohort by cohort; unfortunately we are not able to estimate a complete set of cohort-specific parameters. Consider, for instance, the 2000 cohort: we only have one observation per customer — whether or not each new donor made a repeat donation in 2001 (i.e., $n = 1$) — and as such cannot identify the model parameters. The obvious, albeit possibly restrictive, solution is to pool all six cohorts and estimate a single set of model parameters. We now turn our attention to such an analysis, examining how well the BG/BB model predicts the behavior of the complete group of the 56,847 people who made their first-ever donation to the radio station between 1995 and 2000.

The maximum likelihood estimates of the model parameters are reported in Table 9. (Comparing the fit of the BG/BB model with that of the beta-Bernoulli model, we once again note that the addition of the "death" component results in a major improvement in model fit.) We also note that the BG/BB parameters for the pooled model are remarkably similar to those of the 1995 cohort by itself (Table 4) — this reflects both the high reliability of the model as well as the "poolability" of the cohorts. Figure 7, which compares the expected number of people making $0, 1, \ldots, 6$ repeat transactions between 1996 and 2001 with the observed frequencies, confirms that the model provides a very good fit to the data.

| | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | LL |
|---|---|---|---|---|---|
| BB | 0.501 | 0.753 | | | −115,615.0 |
| BG/BB | 1.188 | 0.749 | 0.626 | 2.331 | −110,521.0 |

**Table 9:** Parameter estimates, pooling the 1995 – 2000 cohorts.

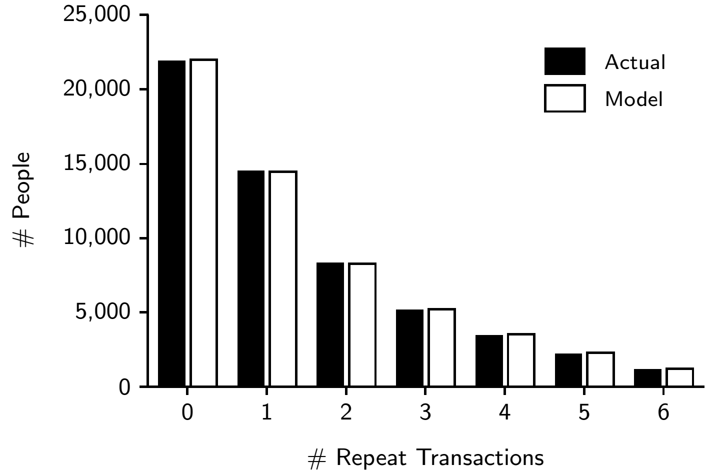



**Figure 7:** Predicted versus actual frequency of repeat transactions by the 1995 − 2000 cohorts.

The pooled model continues to accurately track the actual number of repeat transactions over time. Viewing Figure 8a, which shows the actual vs. predicted cumulative number of repeat transactions, we see that the model overforecasts the holdout transactions by a mere 0.25%. Looking at Figure 8b, which reports these numbers on a year-by-year basis, we note that the BG/BB model clearly captures the underlying trend in repeat transactions. (The repeat transaction numbers rise up to 2001, as new supporters continue to enter the combined pool of donors; after that point, we are focusing on a fixed group of 56,847 potential repeat supporters.) The conditional expectation plots, omitted in the interests of space, are similarly impressive.

This pooled analysis provides a further illustration of the remarkable ability of the BG/BB model to describe and predict the future behavior of a customer base. It is encouraging to see how one set of parameters can capture the behavior of different cohorts acquired across six consecutive years (1995 − 2000) and project their actions quite accurately into the future.

# 4 Discussion

We have developed a new model that can be used to answer standard customer-base analysis questions in noncontractual settings where opportunities for transactions occur at discrete intervals (i.e., a discrete-time analog of the Pareto/NBD model). Using a dataset on annual donations made by the supporters of a public radio station located in the Midwestern United

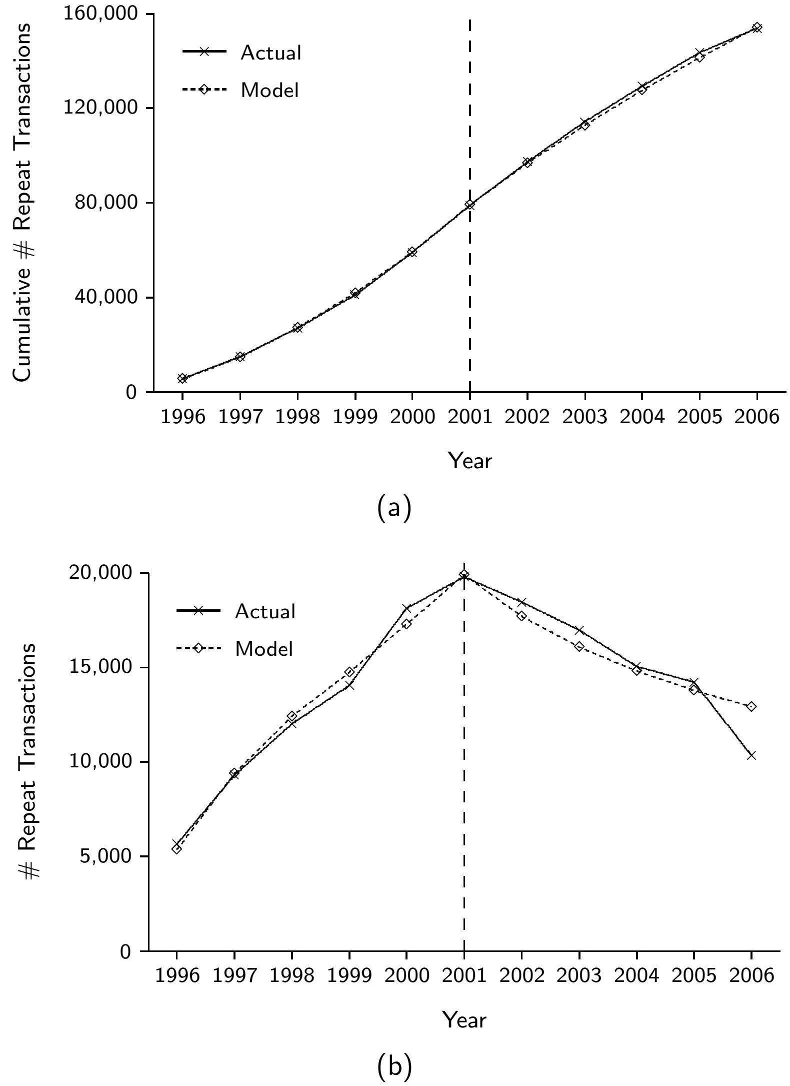



**Figure 8:** Predicted versus actual (a) cumulative and (b) annual repeat transactions.

States, we have demonstrated how the model can be used to compute a number of managerially relevant quantities, such as future purchasing patterns, both collectively and individually (conditional on past behavior). In examining these quantities, we have observed some interesting effects of past behavior (as summarized by recency and frequency) on predictions about future behavior.

It is worthwhile to contrast the BG/BB with the Pareto/NBD model and some of the equivalent analyses performed using it (e.g., in studies such as Fader et al. (2005)). First, as we mentioned right from the outset of this paper, the BG/BB is the direct analog of the Pareto/NBD as one moves from a continuous-time setting to a discrete-time domain. We have brought up a number of specific examples where this distinction is critically important, as well as some situations (characterized as "discretized by recording process" in Figure 1) where the analyst might intentionally convert a continuous-time setting into a discrete-time one primarily to be able to use the BG/BB model instead of the Pareto/NBD. We are aware of several organizations (including hotel chains, financial services firms, and a variety of non-profits) that have chosen this route. The fact that they have done so is an indication of the unique benefits associated with this new model.

These benefits have been mentioned throughout the paper, but we summarize them here:

- The BG/BB offers tremendous advantages in terms of the required data structures. The size of the data summary required for model estimation is purely a function of the number of transaction opportunities — not the number of customers — and therefore the model is highly "scalable" to customer bases of different sizes. Furthermore, in recognizing that recency and frequency are sufficient summary statistics, the relationship between the number of transaction opportunities and the size of the dataset is on the order of $n^2$, which is a significant reduction compared to using the full binary strings (order $2^n$).

- Besides the efficient data requirements, the calculations associated with the model are much simpler than those of the Pareto/NBD. No unconventional or computationally demanding functions are required for parameter estimation or for most of the diagnostic statistics that emerge from the model. Taken together with the aforementioned data advantages, this means that the model is easy to fully implement and utilize within a standard spreadsheet environment. This is very appealing to practitioners, since this reduction in space/effort can be accomplished at virtually no cost (i.e., without sacrificing anything in model performance, as shown in our empirical analyses).

- The discrete nature of the data and the associated behavioral "story" lead to model diagnostics that are convenient to display and are readily interpretable. For instance it is very easy to see and appreciate the non-linear pattern associated with high frequency and low recency, shown in Table 5. Likewise, a simple examination of that table instantly answers the managerial questions raised in the introduction.

- Finally, it is relatively easy to build and analyze the BG/BB model across multiple cohorts of customers — something that has been done rarely (if ever) in the Pareto/NBD literature. Not only does this make the model even more practical, but the multi-year empirical results shown here offer much stronger support for the model's validity than a single-cohort analysis can provide.

While the BG/BB is an excellent starting point for modeling discrete-time noncontractual data, there are several natural extensions worth investigating in future research. First, as is the case with the Pareto/NBD model, the BG/BB model will need to be augmented by a model of purchase amounts when we are interested in the overall monetary value of each customer. A natural candidate would be the gamma-gamma mixture (Colombo and Jiang 1999) that Fader et al. (2005) use in conjunction with the Pareto/NBD model. In situations where the data have been discretized for analysis and reporting purposes, there is the possibility that more than one transaction could occur in each discrete time interval. In such cases, we should derive the monetary-value multiplier by first modeling the number of transactions (conditional on the fact that at least one transaction occurred) and then multiply this by the average value per transaction. A logical model would be the shifted beta-geometric distribution (as used by Morrison and Perry (1970) to model purchase quantity, conditional on purchase incidence).

Second, we may want to allow for a non-zero-order purchasing process at the individual level. A good historical starting point would be the "Brand Loyal Model" (Massy, Montgomery, and Morrison 1970). This would effectively be an extension of Morrison et al.'s (1982) Markov chain model of retail customer behavior at Merrill Lynch; an extension in which the "exit parameter" is allowed to be heterogeneous and is estimated directly from the data (as opposed to being derived from other data sources).

Finally, the model can be extended to include covariate effects, such as customer demographics, marketing actions, and measures of competitive activity. It would be worthwhile to find out which kinds of marketing actions increase buying propensities while the customer is "alive," in contrast to actions that primarily serve to stretch out the customer's probable lifetime. Such a decomposition could have huge implications for developing marketing plans and allocating resources. In bringing in covariates, it will likely be necessary to move away from the simple spreadsheet-based model implementation featured here towards a more complex hierarchical Bayes framework. This would enable additional benefits (such as allowing for correlations between the two underlying behavioral processes) but also comes at a cost (i.e., speed and convenience of model estimation, loss of closed-form expressions for key model-derived measures, and the loss of the simple recency/frequency data structure whose size is independent of the number of people in the sample). But it makes sense to develop such a model and weigh the associated tradeoffs.

But even when these extensions are undertaken, the BG/BB model in its basic form should still serve as an appropriate benchmark model; furthermore, the diagnostic statistics discussed here should continue to be utilized in evaluating the overall performance of the model as well as the incremental value of any additions that might be brought into it. It might be hard for researchers to improve upon the results shown here, but we sincerely encourage the development and testing of richer models to help us gain an even better understanding of customer behavior in the discrete-time, noncontractual setting.

## Appendix

In this appendix we present derivations of the key results presented in Section 2.2. Before starting, we first recall that for $0 < k < 1$,

- the sum of the first $n$ terms of a geometric series is

$$a + ak + ak^2 + \cdots ak^{n-1} = a\frac{1-k^n}{1-k}\,, \qquad \text{(A1)}$$

- the sum of an infinite geometric series is

$$\sum_{n=0}^{\infty} ak^n = \frac{a}{1-k}\,, \qquad \text{(A2)}$$

and note the following transformation of Euler's integral representation of the Gaussian hypergeometric function (${}_2F_1(a, b; c; z)$):

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = B(b, c-b){}_2F_1(a, b; c; z)\,, \quad c > b\,. \qquad \text{(A3)}$$

### A.1 Derivation of (6)

An individual making $x$ purchases had to be alive for at least the first $x$ transactions opportunities. Conditional on $p$, the probability of observing $x$ transactions out of the $i$ (unobserved) transaction opportunities ($i = x, \ldots, n$) the customer is alive is

$$\binom{i}{x} p^x (1-p)^{i-x}\,.$$

Removing the conditioning on being alive for $i$ transaction opportunities by multiplying this by the probability that the individual is alive for that length of time give us

$$P(X(n) = x \,|\, p, \theta) = \binom{n}{x} p^x (1-p)^{n-x}(1-\theta)^n + \sum_{i=x}^{n-1} \binom{i}{x} p^x (1-p)^{i-x}\theta(1-\theta)^i\,. \qquad \text{(A4)}$$

Taking the expectation of this over the mixing distributions for $p$ and $\theta$ ((1) and (2), respectively) gives us (6).

### A.2 Derivation of (7)

Conditional on $p$ and $\theta$, the expected number of transactions over $n$ transaction opportunities is computed as

$$\begin{aligned} E(X(n) \mid p, \theta) &= \sum_{t=1}^{n} P(Y_t = 1 \mid p, \text{alive at } t) P(\text{alive at } t \mid \theta) \\ &= p \sum_{t=1}^{n} (1-\theta)^t \\ &= p(1-\theta) \sum_{s=0}^{n-1} (1-\theta)^s \,, \end{aligned}$$

which, recalling (A1) and performing some further algebra,

$$= \frac{p(1-\theta)}{\theta} - \frac{p(1-\theta)^{n+1}}{\theta} \,. \tag{A5}$$

Taking the expectation of this over the mixing distributions for $p$ and $\theta$ gives us

$$E(X(n) \mid \alpha, \beta, \gamma, \delta) = \left(\frac{\alpha}{\alpha+\beta}\right) \left\{ \frac{B(\gamma-1, \delta+1) - B(\gamma-1, \delta+n+1)}{B(\gamma, \delta)} \right\} .$$

(Strictly speaking, the use of the integral representation of the beta function to solve the integral associated with taking the expectation over $\theta$ only holds for $\gamma > 1$. However it can be shown that we arrive at the same result when $0 < \gamma < 1$.) Representing the beta functions in terms of gamma functions and recalling the recursive property of gamma functions gives us (7).

### A.3 Derivation of (8) and (9)

Recalling (A4), it follows from the memoryless nature of the death process that

$$\begin{aligned} P(X(n, n+n^*) = x^* \mid p, \theta, \text{alive at } n) = &\binom{n^*}{x^*} p^{x^*} (1-p)^{n^*-x^*} (1-\theta)^{n^*} \\ &+ \sum_{i=x^*}^{n^*-1} \binom{i}{x^*} p^{x^*} (1-p)^{i-x^*} \theta (1-\theta)^i \,. \end{aligned} \tag{A6}$$

Noting that the probability that someone is alive at $n$ is $(1-\theta)^n$, we have

$$P(X(n,n+n^*)=x^*\,|\,p,\theta)=\delta_{x^*=0}\Big\{1-(1-\theta)^n\Big\}$$
$$+\binom{n^*}{x^*}p^{x^*}(1-p)^{n^*-x^*}(1-\theta)^{n+n^*}+\sum_{i=x^*}^{n^*-1}\binom{i}{x^*}p^{x^*}(1-p)^{i-x^*}\theta(1-\theta)^{n+i}\,.$$

(The first term accounts for the fact that anyone not alive at $n$ will, by definition, not make any purchases in the interval $(n, n+n^*]$.) Taking the expectation of this over the mixing distributions for $p$ and $\theta$ gives us (8).

By definition, $X(n,n+n^*)=X(n+n^*)-X(n)$; it follows that $E[X(n,n+n^*)]=E[X(n+n^*)]-E[X(n)]$. Substituting (7) in this gives us (9).

### A.4 Derivation of (10)

Reflecting on (4), the first term is the likelihood of $x$ purchases out of $n$ transaction opportunities under the assumption that the customer was alive for all $n$ transaction opportunities. (The other terms account for the possibility that the individual died before $n$.) Using Bayes' theorem, it follows that the probability that a customer with purchase history $(x, t_x, n)$ is alive at $n$ is

$$P(\text{alive at } n\,|\,p,\theta,x,t_x,n)=\frac{p^x(1-p)^{n-x}(1-\theta)^n}{L(p,\theta\,|\,x,t_x,n)}\,. \tag{A7}$$

It follows that

$$P(\text{alive at } n+1\,|\,p,\theta,x,t_x,n)=\frac{p^x(1-p)^{n-x}(1-\theta)^{n+1}}{L(p,\theta\,|\,x,t_x,n)}\,. \tag{A8}$$

By Bayes' theorem, the joint posterior distribution of $p$ and $\theta$ is given by

$$f(p,\theta\,|\,\alpha,\beta,\gamma,\delta,x,t_x,n)=\frac{L(p,\theta\,|\,x,t_x,n)f(p\,|\,\alpha,\beta)f(\theta\,|\,\gamma,\delta)}{L(\alpha,\beta,\gamma,\delta\,|\,x,t_x,n)}\,, \tag{A9}$$

where the individual elements are given in (1), (2), (4), and (5).

Taking the expectation of (A8) over the joint posterior distribution of $p$ and $\theta$, (A9), gives us (10).

By the same logic, we can derive an expression for the probability that a customer with purchase history $(x, t_x, n)$ is alive at transaction opportunity $n+m$. Conditional on $p$ and $\theta$,

$$P(\text{alive at } n+m \,|\, p, \theta, x, t_x, n) = \frac{p^x(1-p)^{n-x}(1-\theta)^{n+m}}{L(p, \theta \,|\, x, t_x, n)}\,.$$

Taking the expectation of this over the joint posterior distribution of $p$ and $\theta$ yields

$$\begin{aligned} &P(\text{alive at } n+m \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n) \\ &\quad = \frac{B(\alpha+x, \beta+n-x)}{B(\alpha, \beta)} \frac{B(\gamma, \delta+n+m)}{B(\gamma, \delta)} \Big/ L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n)\,. \end{aligned} \tag{A10}$$

## A.5 Derivation of (11)

By definition,

$$\begin{aligned} P(X(n, n+n^*) = x^* \,|\, p, \theta, x, t_x, n) &= \delta_{x^*=0}\Big\{1 - P(\text{alive at } n \,|\, p, \theta, x, t_x, n)\Big\} \\ &\quad + P(X(n, n+n^*) = x^* \,|\, p, \theta, \text{alive at } n) P(\text{alive at } n \,|\, p, \theta, x, t_x, n)\,. \end{aligned}$$

Substituting (A6) and (A7) in this, and taking the expectation over the joint posterior distribution of $p$ and $\theta$, (A9), gives us (11).

## A.6 Derivation of (12)

Conditional on $p$ and $\theta$, the expected number of transactions across the next $n^*$ transaction opportunities (i.e., in the interval $(n, n+n^*]$) by a customer with purchase history $(x, t_x, n)$ is

$$\begin{aligned} &E(X(n, n+n^*) \,|\, p, \theta, x, t_x, n) \\ &\quad = E(X(n, n+n^*) \,|\, p, \theta, \text{alive at } n) \times P(\text{alive at } n \,|\, p, \theta, x, t_x, n)\,. \end{aligned}$$

Now

$$\begin{aligned} &E(X(n, n+n^*) \,|\, p, \theta, \text{alive at } n) \\ &\qquad = \sum_{t=n+1}^{n+n^*} P(Y_t = 1 \,|\, p, \text{alive at } t) P(\text{alive at } t \,|\, \theta, t > n) \end{aligned}$$

$$
\begin{aligned}
&= p \sum_{t=n+1}^{n+n^*} \frac{(1-\theta)^t}{(1-\theta)^n} \\
&= p \sum_{s=1}^{n^*} (1-\theta)^s \\
&= \frac{p(1-\theta)}{\theta} - \frac{p(1-\theta)^{n^*+1}}{\theta} \,. \qquad \text{(A11)}
\end{aligned}
$$

Taking the expectation of the product of (A7) and (A11) over the joint posterior distribution of $p$ and $\theta$, (A9), gives us (12).

## A.7 Derivation of (13) and (14)

Integrating (A9) over $\theta$ gives us the marginal posterior distribution of $p$:

$$
f(p \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n) = \mathcal{A} \big/ L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n) \qquad \text{(A12)}
$$

where

$$
\begin{aligned}
\mathcal{A} = {} & \frac{p^{\alpha+x-1}(1-p)^{\beta+n-x-1}}{B(\alpha,\beta)} \frac{B(\gamma,\delta+n)}{B(\gamma,\delta)} \\
& + \sum_{i=0}^{n-t_x-1} \frac{p^{\alpha+x-1}(1-p)^{\beta+t_x-x+i-1}}{B(\alpha,\beta)} \frac{B(\gamma+1,\delta+t_x+i)}{B(\gamma,\delta)} \,.
\end{aligned}
$$

(This equation was used to create the curves in Figure 6a.)

Integrating (A9) over $p$ gives us the marginal posterior distribution of $\theta$:

$$
f(\theta \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n) = \mathcal{B} \big/ L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n) \qquad \text{(A13)}
$$

where

$$
\begin{aligned}
\mathcal{B} = {} & \frac{B(\alpha+x,\beta+n-x)}{B(\alpha,\beta)} \frac{\theta^{\gamma-1}(1-\theta)^{\delta+n-1}}{B(\gamma,\delta)} \\
& + \sum_{i=0}^{n-t_x-1} \frac{B(\alpha+x,\beta+t_x-x+i)}{B(\alpha,\beta)} \frac{\theta^{\gamma}(1-\theta)^{\delta+t_x+i-1}}{B(\gamma,\delta)} \,.
\end{aligned}
$$

(This equation was used to create the curves in Figure 6b.)

We note that when $t_x = n$, (A12) and (A13) both collapse to the following beta distributions:

$$\begin{aligned} f(p \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n) &= \frac{p^{\alpha+x-1}(1-p)^{\beta+n-x-1}}{B(\alpha+x, \beta+n-x)} \\ f(\theta \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n) &= \frac{\theta^{\gamma-1}(1-\theta)^{\delta+n-1}}{B(\gamma, \delta+n)} \,. \end{aligned}$$

In order to derive an expression for the mean of the marginal posterior distribution of $p$, we could solve

$$\int_0^1 p f(p \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n)\, dp \,.$$

However, a simpler derivation is as follows:

$$E(P \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n) = \int_0^1 \int_0^1 p f(p, \theta \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n)\, dp\, d\theta$$

which, recalling (A9)

$$\begin{aligned} &= \int_0^1 \int_0^1 p \frac{L(p, \theta \,|\, x, t_x, n) f(p \,|\, \alpha, \beta) f(\theta \,|\, \gamma, \delta)}{L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n)}\, dp\, d\theta \\ &= \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)} \int_0^1 \int_0^1 \frac{L(p, \theta \,|\, x, t_x, n) f(p \,|\, \alpha+1, \beta) f(\theta \,|\, \gamma, \delta)}{L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n)}\, dp\, d\theta \\ &= \left(\frac{\alpha}{\alpha+\beta}\right) \frac{L(\alpha+1, \beta, \gamma, \delta \,|\, x, t_x, n)}{L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n)} \,, \end{aligned}$$

where $L(\alpha+1, \beta, \gamma, \delta \,|\, x, t_x, n)$ is simply (5) evaluated using $\alpha + 1$ in place of $\alpha$. This is the expression given in (13).

Equivalently,

$$\begin{aligned} E(\Theta \,|\, \alpha, \beta, \gamma, \delta, x, t_x, n) &= \int_0^1 \int_0^1 \theta \frac{L(p, \theta \,|\, x, t_x, n) f(p \,|\, \alpha, \beta) f(\theta \,|\, \gamma, \delta)}{L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n)}\, dp\, d\theta \\ &= \left(\frac{\gamma}{\gamma+\delta}\right) \frac{L(\alpha, \beta, \gamma+1, \delta \,|\, x, t_x, n)}{L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n)} \,, \end{aligned}$$

which is the expression for the mean of the marginal posterior distribution of $\theta$ given in (14).

## A.8 Derivation of (15)

The number of discounted expected residual transactions for a customer alive at $n$ is

$$
\begin{aligned}
DERT(d \mid p, \theta, \text{alive at } n) &= \sum_{t=n+1}^{\infty} \frac{P(Y_t = 1 \mid p, \text{alive at } t) P(\text{alive at } t \mid t > n, \theta)}{(1+d)^{t-n}} \\
&= p \sum_{t=n+1}^{\infty} \frac{(1-\theta)^{t-n}}{(1+d)^{t-n}} \\
&= p \frac{1-\theta}{1+d} \sum_{s=0}^{\infty} \left( \frac{1-\theta}{1+d} \right)^{s},
\end{aligned}
$$

which, recalling (A2),

$$
= \frac{p(1-\theta)}{d+\theta}. \tag{A14}
$$

Multiplying this by the probability that a customer with purchase history $(x, t_x, n)$ (and latent transaction and dropout probabilities $p$ and $\theta$) is still alive at transaction opportunity $n$, (A7), gives us

$$
DERT(d \mid p, \theta, x, t_x, n) = \frac{p^{x+1}(1-p)^{n-x}(1-\theta)^{n+1}}{(d+\theta) L(p, \theta \mid x, t_x, n)}. \tag{A15}
$$

Taking the expectation of this over the joint posterior distribution of $p$ and $\theta$, (A9), gives us

$$
DERT(d \mid \alpha, \beta, \gamma, \delta, x, t_x, n) = \mathcal{D} \times \frac{B(\alpha + x + 1, \beta + n - x)}{B(\alpha, \beta)} \Big/ L(\alpha, \beta, \gamma, \delta \mid x, t_x, n),
$$

where

$$
\mathcal{D} = \int_0^1 \frac{1}{d+\theta} \frac{\theta^{\gamma-1}(1-\theta)^{\delta+n}}{B(\gamma, \delta)} \, d\theta
$$

letting $s = 1 - \theta$

$$
\begin{aligned}
&= \frac{1}{B(\gamma, \delta)} \int_0^1 \frac{1}{(1+d) - s} (1-s)^{\gamma-1} s^{\delta+n} \, ds \\
&= \frac{1}{B(\gamma, \delta)(1+d)} \int_0^1 s^{\delta+n} (1-s)^{\gamma-1} \big(1 - \big(\tfrac{1}{1+d}\big) s\big)^{-1} \, ds
\end{aligned}
$$

which, recalling (A3)

$$= \frac{B(\gamma, \delta + n + 1)}{B(\gamma, \delta)(1 + d)} \, {}_2F_1\big(1, \delta + n + 1; \gamma + \delta + n + 1; \tfrac{1}{1+d}\big)\,,$$

giving us the expression in (15).

Since $L(\alpha, \beta, \gamma, \delta \,|\, x, t_x, n) = 1$ when $x = t_x = n = 0$, it follows that the number of discounted expected transactions for a just-acquired customer is

$$DET(d \,|\, \alpha, \beta, \gamma, \delta) = \left(\frac{\alpha}{\alpha + \beta}\right)\left(\frac{\delta}{\gamma + \delta}\right) \frac{{}_2F_1\big(1, \delta + 1; \gamma + \delta + 1; \tfrac{1}{1+d}\big)}{1 + d}\,. \tag{A16}$$

To compute $DET$ for an yet-to-be-acquired customer, we need to add 1 to this quantity (i.e., the purchase at time $t = 0$ that corresponds to the customer's first-ever purchase with the firm and therefore starts the transaction opportunity clock).

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