Frailty Correlated Default

Darrell Duffie, Andreas Eckner, Guillaume Horel, Leandro Saita

Stanford University

Preliminary

Moodys-NYU Credit Risk Conference
New York, May, 2006

Stanford University, 2006
Figure 1: Aggregate default intensities and number of defaults, 1979-2004.
Default Intensity

- Default intensity: the conditional mean arrival rate of default, in events per year.

- Example: An intensity of 0.2% is a mean arrival rate of 2 defaults per 1000 obligor-years, or roughly a probability of default of 0.2% per year.

- We estimate the parameter vector $\beta$ for default intensity of the form $\Lambda(X_t; \beta)$, where $X_t$ is a list of covariates, including the firm’s leverage, volatility, other firm-specific and macro-economic covariates, and “frailty” factors, both common and firm-specific.
Dynamic Default Frailty Model

Firm $i$ default intensity:

$$\lambda_{it} = e^{\alpha + \beta \cdot W_{it} + Y_t + Z_i}$$

where

- $W_{it}$ is a vector of observable covariates.
- $Y$ is a common unobserved Brownian motion.
- and $Z_i$ is unobserved independent-gamma heterogeneity.
Matchable U.S. Industrials 1979-2004

- 402,000 firm-months of data.
- 2,793 firms.
- 496 defaults.
- 1,072 exits by merger or acquisition.
Some Related Work


- Beaver (1968), Altman (1968), ..., Shumway (2001), Beaver, McNichols, and Rhie (2004), Duffie, Saita, Wang (2005), Campbell, Hilscher, and Szilagyi (2005), ...

Key covariate: distance to default

Asset and Liability Values

Stanford University, 2006
Figure 3: The dependence of empirical default frequency on distance to default. Source: Duffie, Saita, Wang, *J. Financial Economics*, 2006.
Figure 4: Posterior of the frailty factor, conditioning in 2004.
Figure 5: The frailty distribution, conditioning “as you go.”
Figure 6: Posterior of the frailty factor in January 1998, conditioning in 2004 (blue), and conditioning in January 1998 (red).
Figure 7: Term structure of hazard rates, Xerox, December 2003: frailty (blue), no frailty (red), unobserved heterogeneity (green).
Figure 8: Default correlation of ICO and Xerox with frailty (blue), without frailty (red), and with frailty and unobserved heterogeneity (green).
Figure 9: 5-year portfolio defaults: common frailty (blue), independent frailty (red), common start only (green).
Test of Doubly Stochastic Model


- Doubly-stochastic: Defaults are independent conditional on the factors determining default intensities.
- Implication: Re-scaling time, with “new time” passing at rate \( \sum_i \lambda_i(t) \) per unit of calendar time, total default arrivals are Poisson at rate parameter 1.
Joint Likelihood with Frailty

• The joint covariate process \( X = (W, Y) \), with \( W \) observed, \( Y \) independent and not observed, is a Markov process whose transition distribution is determined by a parameter vector \( \gamma \).

• By Bayes’ Rule, with observable \( X \), the joint likelihood of \( X \) and the censored default times \( \tau = (\tau_1, \ldots, \tau_n) \), is

\[
\mathcal{L}(\tau, X; \beta, \gamma) = \mathcal{L}(\tau \mid X; \beta) \times \mathcal{L}(X; \gamma).
\]

• With unobservable \( Y \),

\[
\mathcal{L}(\tau, W; \beta, \gamma) = E \left[ \mathcal{L}(\tau \mid Y, W; \beta) \mid W \right] \times \mathcal{L}(W; \gamma),
\]

evaluated by Gibbs sampling.
Figure 14: Annual default rates, investment (IG) and speculative (SG). Source: Moodys (2005).
Default Intensity Estimates

$$\lambda(t) = e^{\beta_0 + \beta_1 X_1(t) + \cdots + \beta_n X_n(t)}$$

Maximum Likelihood estimate of $\beta_i$ (with standard error)

- Distance to default $-1.13$ (0.036)
- Trailing 1-year stock return $-0.694$ (0.075)
- 3-month T-Bill rate $-0.105$ (0.021)
- Trailing 1-year SP500 return $1.20$ (0.289)
Conditional Likelihood Function

• For a given firm \( i \), the \( X \)-conditional likelihood of survival from \( t(i) \) to \( \tau_i \), censored at \( T(i) \), is

\[
\mathcal{L}_i(\tau_i \mid X; \beta) = q_{t(i)} q_{t(i)+1} \cdots q_{T(i)-1}, \quad \text{if } \tau_i > T(i),
\]

\[
= q_{t(i)} q_{t(i)+1} \cdots q_{\tau_i-2} (1 - q_{\tau_i-1}), \quad \text{if } \tau_i \leq T(i),
\]

where \( q_t = e^{-\Lambda(X_{it}; \beta)} \) is the one-period \( X \)-conditional survival probability.

• The doubly-stochastic assumption implies that the \( X \)-conditional likelihood of \( \tau_1, \ldots, \tau_n \), as censored, is

\[
\mathcal{L}(\tau \mid X; \beta) = \prod_i \mathcal{L}_i(\tau_i \mid X; \beta).
\]