A Queueing Network Approach to Portfolio Credit Risk

Mark Davis and Juan C. Esparragoza
Department of Mathematics
Imperial College, London

31st October 2004

1 Introduction

In the modelling of credit risk it has been always important to consider a portfolio as a whole. Moreover, both risk management and valuation has been challenged by the development of the credit risk market with instruments based on baskets such as first to default swaps and CDOs.

Modelling the credit risk of a portfolio involve:

- Credible Modelling of Interaction Effects: The literature distinguish between two main interaction effects. First, common factors that influence defaults, such as the economic cycle. Second, the contagion effect where the default of an obligor may affect the probability of default of others obligors.

- Efficient Computational Methods: Many models that are convenient for a single obligor or a few of them may be computationally expensive to use for large portfolios. For example, the use of reduced form models with copula dependence structure can be cumbersome when the number of obligors increases. Algorithms of simulation/pricing must show little sensitivity in computational terms to the number of elements.

- Ease of Calibration: Together with the computational issue, a desirable property of a model is an easy calibration. It is reflected in the existence f a methodology that allows for simplicity . A parsimony with the number of parameters must be reached. While a large number of parameters may be cumbersome and redundant, too few may prove insufficient to reflect all the market data.

In the case of large portfolios it makes sense to consider the use of large-sample approximations in order to achieve efficient computational algorithms and keep easy calibrations.

Maybe the best known work that illustrates the modelling of large portfolios by using an homogeneity assumption assumption is by Vasicek (1987). We briefly present below this model to introduce our proposal. More recent work by Hull and White (2004) extends this to non-homogeneous portfolios. The authors present an algorithm to simplify the computational burden of the model. Unfortunately, sometimes this proves to be unstable and some modifications are required.

Another model that is widely used and exploits the use of approximations is CreditRisk+ (CSFB 1997). Recently, Gordy (2001) proposed the use of saddlepoint approximations to calculate quantiles of the loss distribution under CreditRisk+ outperforming the commonly used Panjer recursion method.

A different approach to the loss distribution estimation is given by Giesecke and Weber (2003) applying a gaussian approximation to the loss distribution in a voter model.

All models mentioned above try to estimate the loss distribution of a portfolio in a fixed point in the time, this is, they are static. Recently, Frey and Backhaus (2003) presented a model similar to the one presented here and considering a mean field interaction approach. They derive strong convergence results conditional on a general exogenous process.

This paper is structured as follows. In order to introduce the idea of large portfolio modelling we review briefly the Vasicek model and comment on some of its advantages and critical points. In section 3 we introduce heuristically the ideas and assumptions of the proposed model, in particular we intend to present the queuing network as a natural structure for the credit risk modelling. The general model is presented formally in section 4 together with the results that allow approximate the distribution of the process. Section 5 presents the application to a simple credit risk model and explicit solutions to the approximated processes are presented. Finally we present some numerical results in section 6 and conclude.

2 The Vasicek Large Portfolio Model

The single factor gaussian copula model proposed by Vasicek (1987) is maybe the most popular framework used to price and quote portfolio basket related derivatives. The use of the model is wide in the CDO tranche pricing due to its simplicity.

Given a portfolio with n obligors labeled $i = 1, .., n$, the obligor i defaults if $X_i < K_i$ where X_i is a standard normal random variable and $K_i = N^{-1}(p_i)$ is a barrier defined to match p_i the marginal default probability of the obligor i. Represent X_i as

$$X_i = \rho_i X + \sqrt{1 - \rho_i^2}\epsilon_i,$$

where $X, \epsilon_1, \epsilon_2, \ldots$ are independent standard normal random variables. Using the homogeneity assumptions $p_i = p$ and $\rho_i = \rho$, $i = 1, \ldots, n$, the conditional marginal probability of default for each obligor i is

$$P[\text{Obligor } i \text{ defaults}|X] = N\left(\frac{K - \rho X}{\sqrt{1-\rho^2}}\right) \equiv p(X),$$

where $K = N^{-1}(p)$. From the construction the defaults are independent conditional on X. The conditional distribution of *proportion* π of obligors defaulting is then binomial with mean $p(X)$ and standard deviation $\sqrt{p(X)(1-p(X)/n}$ where n is the portfolio size.

By numerical integration it is possible to calculate the unconditional loss distribution

$$P[\pi = a] = \left(\begin{array}{c} n \\ k \end{array}\right) \int_{-\infty}^{\infty} p^a(x)(1-p(x))^{n-a}\phi(x)dx$$

where $\phi(x)$ denotes the normal density.

Moreover, for large n the standard deviation is small and we have approximately

$$(\pi > \alpha) \Leftrightarrow p(X) > \alpha$$

giving the *unconditional* distribution

$$P[\pi > \alpha] \sim N\left(\frac{K_1 - \sqrt{1-\rho^2}N^{-1}(\alpha)}{\rho}\right).$$

The model as described is static, useful only for valuations in a single point of the future. Extensions of the copula approach to different copula structures and multi-stage valuation points for CDOs and baskets defaults derivatives have been developed, for a survey consult Laurent and Gregory (2004). However, there is no interpretation to the variable X and the parameter ρ is unobservable in the market.

Other criticism to the model is the inability to reflect the quoted prices for CDOs tranches producing the "correlation skew". Using a single correlation parameters seems to undervalue equity and senior tranches while producing the opposite effect to the mezzanine tranches.

Our objective is create a modelling framework in a dynamic context keeping the simplicity that makes the Vasicek model appealing. By introducing a larger number of parameters, we expect to be able to reflect the correlation skew observed in the market.

3 Using Queuing Networks

Rating based models have been widely used for pricing and risk management purposes. Maybe the best known of all of them is the Jarrow, Lando

and Turnbull model (1997). Most recent application of rating models in the context of Markov chains can be found in, for example, Frydman and Schuermann (2004) and Lando and Christensen (2004).

Even if there are some criticisms of the use of credit ratings for pricing purposes, it is appealing to consider having buckets of obligors with similar characteristics to simplify the mathematics while modelling the likelihood of obligors to modify their credit worthiness. This is particularly useful when handling large portfolios.

The use of latent variables and common factors has been common in practice when dealing with large portfolios. Examples well known are the CreditMetrics model (Gupton et al. 1987) and PortfolioRisk+ (CSFB 1997). Moreover, empirical evidence (Crowder, Giampieri and Davis 2003) suggests that the pattern of realized defaults is well represented by a latent variable model where the latent process X_t is a 2-state (good times / bad times) economic variable.

Finally, taking an approach of ratings and latent variables related with economic interpretation, it is sensible to assume that obligors move around rating categories at a faster time scale than the economic cycle.

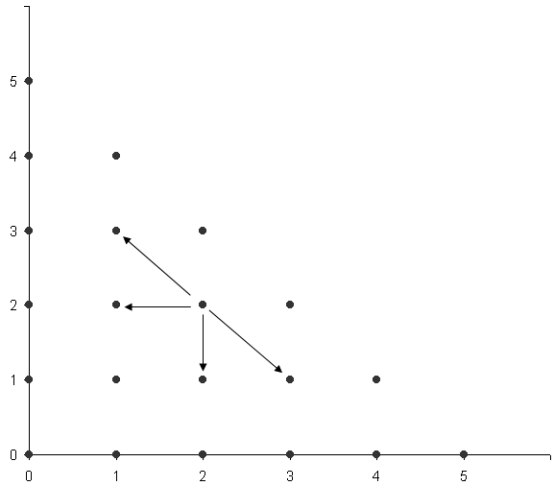
These facts suggest a model in which obligors move around the rating categories at rates depending on the latent process and occasionally default. Moreover, there is an obvious analogy with queueing networks in which 'jobs' move around 'service stations' for processing.

Recent work by Choudhury, Mandelbaum et al. (1998) studies fluid and diffusion limits for queueing networks under random environments. We give some modifications of their results in this paper, in a form suitable for application to credit portfolios.

3.1 Rating Transitions, No Latent Variable and the Leaky Bucket Analysis

For a portfolio with n obligors and K possible ratings $k = 1, ..., K$ we define a vector process $Q^n(t)$ taking values in $\mathbf{N}^k$, with each component $Q^n_k(t)$ containing the number of elements in each rating category at time t. Then, $\sum Q^n_k(t)$ is the number of non defaulted obligors at time t. When $n = 5, K = 2$, the state space of $Q^n(t)$ is as shown in figure 1. The figure shows all the possible movements given the current credit ratings: transitions (move along the diagonal) and defaults (move to the next diagonal).

Figure 1. The Sample Space in the Simplest Case



We can analyse heuristically the model by using a stationary argument applied to the case of two credit ratings, 0 and 1, as in the figure 2. The transition rate rate from 0 to 1 and viceversa are μ_0 and μ_1 respectively, while the default rate in state 0 (1) is ν_0 (ν_1). We call this the *leaky bucket analysis*. Two buckets (0 and 1) are leaking at rating ν_i, $i = 0, 1$. Additionally, liquid goes form one bucket to the other at a rate μ_i. If there is no leaking in the system, we expect that π_t, the proportion of obigors in category 0 (bucket 0) will reach a stationary state

$$\pi = \frac{\mu_1}{\mu_0 + \mu_1}$$

solution to the mass balance equation

$$\mu_0 \pi - \mu_1 (1 - \pi) = 0.$$

Assuming that $\nu_0, \nu_1 << \mu_0, \mu_1$ we anticipate an average leakage (default) rate

$$\nu = \nu_0 \pi + \nu_1 (1 - \pi) = \frac{\nu_0 \mu_1 + \nu_1 \mu_0}{\mu_0 + \mu_1}$$

Moreover, since the obligors are independent in this model, the standard deviation of cumulative defaults in $[0, t]$ is just

$$\sqrt{\frac{d}{n} t}$$

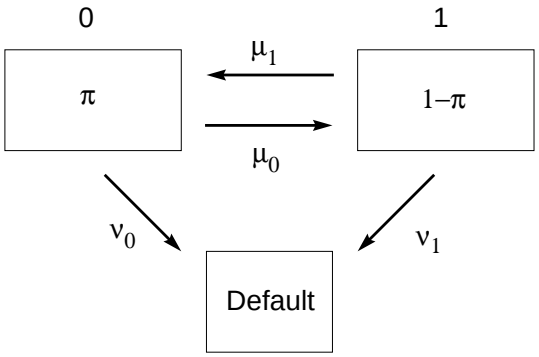
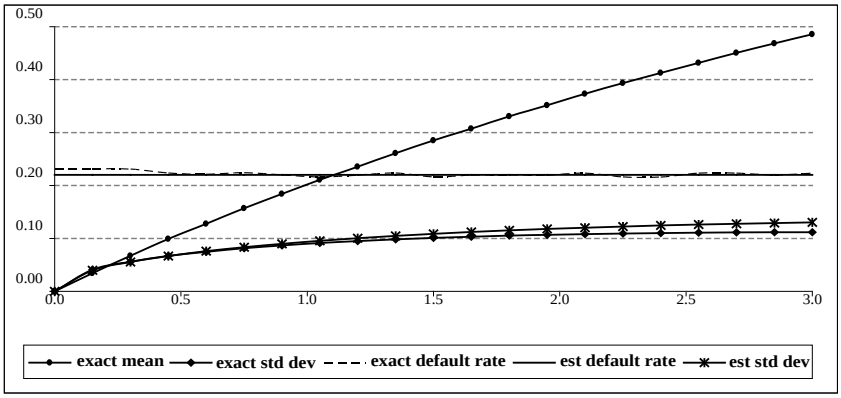


Figure 2. The Leaky Bucket System



parameters

μ_0	4.0	μ_1	1.0
ν_0	0.3	ν_1	0.2
n	20	π_0	0.75

Figure 3. Results of the Leaky Bucket Analysis

As illustrated in figure 3, this simple analysis is surprisingly succesful in predicting the mean an variance of the exact distribution obtained by solving the forward equation for the finite-state Markov process, if an adjustment for mean defaults is made.

However the leaky bucket analysis does not depend on the initial distribution of the obligors' ratings and therefore does not capture the short-term default behavior. As seen in figure 3, the success of the analysis will depend on reaching the stationary state (assuming it exists) and on the relation of leaking and transition rates.

3.2 Random Environments and Fluid and Diffusion Limits

As argued above, the economic cycle is relevant in the default rates of the obligors. We propose a finite state environment process X_t to reflect this effect.

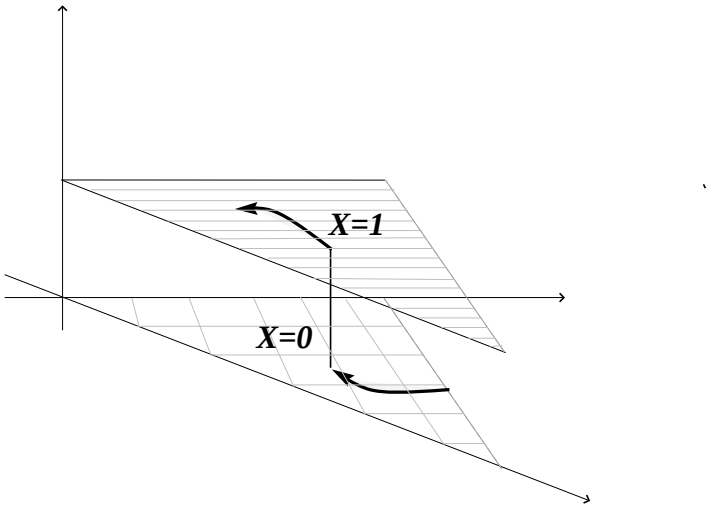


Figure 4. Multiple layer setting

The finite state environment process defines different "layers" in which transition parameters are different. These multiple layers are copies of the initial state space shown in Figure 1. Now the process jumps across the layers anytime the random environment process X_t jumps to a different state as illustrated in figure 4. In this model the movements across the grid are expected to be more frequent than the movements across layers, particularly when the number of elements in the system increases. This is why in some literature this structure is referred as a slowly changing environment (Mandelbaum, Massey et al. 1988). All we can observe however is the second coordinate of the pair process $(X_t, Q^n(t))$.

A more accurate analysis of the model that keeps some of the tractability of the leaky bucket analysis can be done using fluid and diffusion limits. Conditional in the realisation of the random environment, the process $Q^n(t)$ may be approximated by two processes

$$Q^n(t) \simeq Q^{(0)}(t) + \sqrt{n}Q^{(1)}(t)$$

where $Q^{(0)}$ is a deterministic process called the "fluid limit" and $Q^{(1)}(t)$ is a diffusion called the "diffusion limit" of the sequence $Q^n(t)$. This is, conditional on the random environment, the distribution of the process $Q^n(t)$, at any time $t > 0$, may be approximated by a normal distribution. Fluid and Diffusion limits are sometimes referred as Functional Law of Large Numbers and Functional Central Limit Theorem respectively. Now we state our general model.

4 The General Model

The random environment process $X(t)$ is a finite state process in continuous time, having at most a finite number of jumps in any bounded interval of $[0, \infty)$.

To construct the process $Q(t)$ we consider a collection of mutually independent Poisson processes $\{A_i\}_{i \in I = \{1,...,n\}}$ and a collection of vectors $\{\mathbf{v}_i\}$ in R^K, $K \in \mathrm{N}$, and a collection of non-negative functions of the form $\alpha_i(\cdot, \cdot, x) : [0, \infty) \times \mathrm{R}^K \to [0, \infty)$ for all $i \in I$ and $x \in \mathcal{X}$. We assume each $\alpha_i(t, \cdot, x)$ is Lipschitz bounded with respect to the second argument, this is, exist a locally integrable function $\beta_t : [0, \infty) \to [0, \infty)$ such that

$$\|\alpha_i(t, \cdot, x)\| \leq \beta_t \tag{1}$$

where $\| \cdot \|$ is the Lipschitz norm defined as

$$\|f\| = \sup_{\substack{x \neq y \\ x, y \in D_1}} \frac{|f(x) - f(y)|_{D_2}}{|x - y|_{D_1}} \vee |f(0)|_{D_2}$$

We define the probability space $(\Omega^1, \mathcal{F}^1, P^1)$ associated to the random environment $X(t)$, where $\Omega^1 = D([0, \infty), \mathcal{X})$ the space of Right Continuous with Left Limits (RCLL) functions from $[0, \infty)$ into $\mathcal{X}$, $\mathcal{F}^1$ is a σ-algebra in Ω^1 and P^1 is some probability measure. The measure space associated to the collection $\{A_i\}$is defined by the state space $\Omega^2 = \Omega_1 \times ... \times \Omega_n$ where $\Omega_i = D([0, \infty), \mathrm{R}^m)$ and the minimum sigma-algebra $\mathcal{F}^2$ generated by Ω^2. The probability measure P^2 is defined as the product measure. Finally, we assume independence between the environment process and each Poisson process A_i, therefore the probability space is defined as $(\Omega, \mathcal{F}, P)$ with $\Omega = \Omega^1 \times \Omega^2$, $\mathcal{F} = \sigma(\Omega)$ and $P = P^1 \times P^2$. We will denote by $\omega = (\omega_1, \omega_2)$ the elements of Ω being $\omega_1 \in \Omega^1$ and $\omega_2 \in \Omega^2$.

We define a mapping $\mathcal{Q}$ from Ω into $D([0, \infty), \mathrm{R}^K)$ by $(\omega_1, \omega_2) \mapsto Q$, where Q is the process solution to the equation

$$Q(t) = Q(0) + \sum_{i=1}^{n} A_i \left(\int_0^t \alpha_i(s, Q(s), X(s)) ds \right) \mathbf{v}_i \tag{2}$$

The law of $Q(t)$ is $P_Q(\cdot) = P(\mathcal{Q}^{-1}(\cdot))$. The probability conditional on the environment process $X(t)$ denoted by $P_Q^{\omega_1}$ is then defined by the conditional probability under the inverse mapping.

We are concerned with the convergence of sequences $Q^\eta : (\omega^1, \omega^2) \to \mathrm{R}^K$ of the form

$$\frac{Q^\eta(t)}{\eta} = \frac{Q^\eta(0)}{\eta} + \frac{1}{\eta} \sum_{i=1}^{n} A_i \left(\eta \int_0^t \alpha_i(s, Q^\eta(s), X(s)) ds \right) \mathbf{v}_i$$

for $\eta > 0$.

Theorem 1 *Assume a collection of functions of the form* $\alpha_i : [0,\infty) \times \mathrm{R}^K \times \mathcal{X} \to [0,\infty)$ *for* $i \in I$ *and such that*

$$|\alpha_i(t,y,x)| \leq C(1+|y|) \tag{3}$$

for some constant $C < \infty$, $s \geq 0$ *and* $y \in \mathrm{R}^K$. *Assume the processes* $X(t)$ *and* $A_i(t)$ *defined as above and* $\mathbf{v}_i$ *a collection of vectors in* R^K.

Then the process $Q(t)$ *defined as the solution of the equation*

$$Q(t) = Q(0) + \sum_{i=1}^{n} A_i \left(\int_0^t \alpha_i(s, Q(s), X(s)) ds \right) \mathbf{v}_i \tag{4}$$

has unique solution for a.e. ω.

We want to infer the behaviour of the state of the network when the number of arrivals and the number of serves per node increases while the rate of service remains unchanged. We will present some *quenched* convergence results for the process (X_t, Q_t). The results obtained are two: Firstly, a quenched strong approximation limit for *accelerated sequences* of the form $(X(t), Q(\eta t)/\eta)$, $\eta \to \infty$. Secondly, a quenched weak convergence result states the convergence of accelerated sequences of the form

$$\frac{1}{\sqrt{\eta}}(Q(\eta t) - \eta Q^0(t))$$

to a diffusion for a given realisation of the random process. Here $Q^{(0)}$ is the limit derived from the strong approximation.

4.1 Strong Approximations

We define a sequence of network processes $\{(X(t), Q^\eta(t)/\eta); \eta > 0\}$ associated to $(X(t), Q(t))$ as the set of network processes where $Q^\eta(t)$ is the solution to the system

$$Q^\eta(t) = Q^\eta(0) + \sum_{i=1}^{N} A_i \left(\int_0^t \alpha_i^\eta \left(s, Q^\eta(s), X(s)\right) ds \right) \mathbf{v}_i \tag{5}$$

where $\{\alpha_i^\eta(s,\cdot,x)\}$, with $x \in \mathcal{X}$ and $i \in I$, is a collection of functions satisfying

$$\|\alpha_i^\eta(t,\cdot,x)\| \leq \eta \beta_t \tag{6}$$

with β_t a locally integrable function.

The interpretation of the pair $(X(t), 1/\eta Q^\eta(t))$ for some $\eta > 0$ is a process with the same characteristics of $Q(t)$ under the same environment $X(t)$ but where the number of servers and rates of arrivals have increased η times.

A quenched law of large numbers for the sequence $(X(t), 1/\eta Q^\eta(t))$:

Theorem 2 *If* $\{\alpha_i^\eta | i \in I, \eta > 0\}$ *are Lipschitz bounded and*

$$\lim_{\eta\to\infty} \sum_{i=0}^{N} \int_0^t \left\| \frac{\alpha_i^\eta(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x) \right\| ds = 0 \tag{7}$$

For $\omega_1 \in \Omega^1$ *where the process* $Q^\eta(t)$ *is that*

$$\frac{Q^\eta(0)}{\eta} \to Q^{(0)}(0) \ \text{ as } \ \eta \to \infty \tag{8}$$

then

$$\frac{Q^\eta(t)}{\eta} \to Q^{(0)}(t) \ \text{ as } \ \eta \to \infty \tag{9}$$

a.s. in $P_Q^{\omega_1}$*, where* $Q^{(0)}(t)$ *defined in* R^K *is the solution to the equation*

$$Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \alpha^{(0)}(s, Q^{(0)}(s), X(s))ds \tag{10}$$

where

$$\alpha^{(0)}(t,\cdot,x) = \sum_{i=1}^{N} \alpha_i^{(0)}(t,\cdot,x)\mathbf{v}_i \tag{11}$$

and the integral equation is deterministic conditional on the random environment process $X(t)$*.*

The process $Q^{(0)}(t)$ is referred as the *fluid limit* of the sequence $\{Q^\eta(t)\}_{\eta\geq 0}$.

4.2 Weak convergence

It is possible to derive a quenched functional version of the central limit theorem for $(X(t), (1/\eta)Q^\eta(t))$ conditional on $\mathcal{F}_t^{\omega_1}$.

To present this result, we require to define the *scalable Lipschitz derivative* of a function $f : D_1 \to D_2$ at $x \in D_1$, D_1 and D_2 two Banach spaces, as the function $\Lambda f_x(y) : D_1 \to D_2$ such that

$$\lim_{y\to 0} \frac{|f(x+y) - f(x) - \Lambda f_x(y)|_{D_2}}{|y|_{D_1}} = 0$$

whenever such function exists and it is Lipschitz bounded and homogeneous, this is,

$$\|\Lambda f_x(\cdot)\| < \infty,$$

and for all $\lambda \geq 0$

$$\lambda \Lambda f_x(y) = \Lambda f_x(\lambda y).$$

We note that in the case of $f : \mathrm{R}^{d_1} \to \mathrm{R}^{d_2}$, $d_1, d_2 \in \mathrm{N}$, the operator $\Lambda f_x(y)$ generalises the notion of directional derivative, and whenever the differential operator exists we have

$$\Lambda f_x(y) = Df(x)y$$

where $Df(x)$ is the Jacobian matrix valuated at x. When the Jacobian matrix is not defined $\Lambda f_x(y)$ may be not unique.

For a sequence of r.v.'s defined in a measurable space $(\Omega, \mathcal{F})$ we will say that a sequence of r.v.'s $\{Y_n\}$ converges in distribution of Y, denoted by $\lim_{n\to\infty} Y_n =^d Y$, w.r.t. $\mathbf{P}$ some probability measure on $(\Omega, \mathcal{F})$ if for all $f \in \overline{C}(\Omega)$, the set of bounded continuous real functions defined in Ω, $\lim_{n\to\infty} E_P\{Y_n\} = E_P\{Y\}$.

Some weak convergence results in queuing theory assume a heavy traffic condition, this is, the arrival and total service rates are nearly the same. Here we have no arrivals and eventually $Q^\eta(t) = \mathbf{0}$, so no stationarity may be assumed.

Theorem 3 *Assume*

$$\sum_{i\in I} \int_0^t \overline{\lim_{\eta\to\infty}} \sqrt{\eta} \left\| \frac{\alpha_i^\eta(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x) \right\| ds < \infty \tag{12}$$

and

$$\lim_{\eta\to\infty} \sum_{i\in I} \int_0^t \left\| \sqrt{\eta} \left[\frac{\alpha_i^\eta(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x) \right] - \alpha_i^{(1)}(s,\cdot,x) \right\| ds = 0 \tag{13}$$

and for all $x \in \mathcal{X}$ and $i \in I$. Assume the function $\alpha^{(0)}(s,\cdot,x)$ has scalable Lipschitz derivative for any values $X(t)$ and $Q^{(0)}(t)$. For P^1 a.e. $\omega_1 \in \Omega^1$, if

$$\lim_{\eta\to\infty} \sqrt{\eta} \left[\frac{Q^\eta(0)}{\eta} - Q^{(0)}(0) \right] =^d Q^{(1)}(0) \tag{14}$$

w.r.t $P_Q^{\omega_1}$, then

$$\lim_{\eta\to\infty} \sqrt{\eta} \left[\frac{Q^\eta(t)}{\eta} - Q^{(0)}(t) \right] =^d Q^{(1)}(t) \tag{15}$$

w.r.t $P_Q^{\omega_1}$, where the process $Q^{(1)}(t)$ takes values in Ω^2 and it is the solution to the stochastic integral equation

$$\begin{aligned} Q^{(1)}(t) &= Q^{(1)}(0) + \int_0^t \Lambda\alpha_i^{(0)}(s, X_s, Q^{(0)}; Q^{(1)}(s)) + \alpha^{(1)}(s, Q^{(0)}(s), X_s) ds \\ &\quad + \sum_{i\in I} B_i \left(\int_0^t \alpha_i^{(0)}(s, Q^{(0)}(s), X_s) ds \right) \mathbf{v}_i \end{aligned} \tag{16}$$

where $\{B_i(t)\}$ is a collection of independent standard Brownian motions in $(\Omega^2, \mathcal{F}^2, P^{\omega_1})$.

5 Application to Correlated Defaults

We propose a model for the default/rating process of a set of obligors. There is a finite-state random environment process $\{X(t)|t \geq 0\}$ representing some macroeconomic (or sector associated) process that influences the default/transition rates of the obligors. The obligor credit events are independent conditional on the realisation of the environment process and follow a Markov chain with rates being function of the environment process.

Assume a portfolio with n obligors and K possible ratings $1, \ldots, K$. The initial rating composition of the portfolio is represented by the *rating distribution vector* $Q^n(0) \in \mathrm{R}^k$. $Q^n(t)$ will represent the random rating distribution of the portfolio at a later time t.

We define the index set of transition events $I = \{(i,j)|i,j = 1,...,k\}$ and denote the transition rate from rating i to rate j, $i,j = 1,...,k$, by $\mu_{(i,j)}(x) = \mu_{ij}(x)$; default rates are denoted by $\mu_{(i,i)}(x) \equiv \mu_i(x)$. Associated to these credit events we define the set of vectors $\{\mathbf{v}_{(i,j)} \in \mathrm{R}^k|(i,j) \in I\}$ that define the changes in the rating distribution vector in case of a credit event. This is

$$v_{(i,j)} = \begin{cases} \mathbf{e}_j - \mathbf{e}_i & i \neq i \\ -\mathbf{e}_i & i = j \end{cases}$$

where $\mathbf{e}_i$ is the i-th canonical vector in R^k.

Under this assumptions the credit events of each obligors are identically distributed and occur according to the first jump of a Poisson process with rates

$$\hat{\mu}_i(x) = \sum_{j=1}^{k} \mu_{ij}(x)$$

Once a credit event occurs at time t, the obligor defaults with probability $\mu_i(X_t)/\hat{\mu}_i(X_t)$ while a transition to rate $j \neq i$ has probability $\mu_{ij}(X_t)/\hat{\mu}_i(X_t)$.

As stated above, the occurrence of a credit event for an obligor in the credit rate i is given by the first jump of a Poisson process. The rate of occurrence depends on the environment random process but is independent of time. This is, for a set of positive real numbers $\{\mu_{ij}^x|x \in \mathcal{X}, (i,j) \in I\}$ we define the transition default rates as

$$\mu_{(i,j)}(t, X_t) \quad = \quad \sum_{x \in \mathcal{X}} \mu_{ij}^x \mathrm{I}_{\{X_t = x\}}$$

for $(i,j) \in I$.

5.1 The Fluid Limit

According to the notation used above, we have the set of rate functions

$$\alpha_{(i,j)}^n(t, \mathbf{y}, x) = \mathbf{y}_i \mu_{(i,j)}(t, x)$$

and since the rate function does not depend on n is obvious that

$$\alpha^{(0)}_{(i,j)}(t, \mathbf{y}, x) = \mathbf{y}_i \mu_{(i,j)}(t, x)$$

satisfy the conditions of theorem 2. Using vector notation

$$\alpha^{(0)}(t, \mathbf{y}, x) = A_t(x)\mathbf{y}$$

where A is the infinitesimal generator of the process.

We can verify that by assuming

$$\lim \frac{1}{n} Q^n(0) = Q^{(0)}(0)$$

for all $n \geq 0$ all conditions of therorem 2 hold and the fluid limit process $Q^{(0)}(t)$ is the solution to the deterministic PDE system

$$\frac{d}{dt} Q^{(0)}(t) = A_t Q^{(0)}(t) \tag{17}$$

and whenever $A = A_t$ is time independent (no environment influence) the solution is given by

$$Q^{(0)}(t) = e^{tA} Q^{(0)}(t)$$

otherwise, since X_t has at most finite jumps in any bounded interval of time, we can define a countable set of jump times of X_t $t_0 = 0, t_1 < ...$ and define $Q^{(0)}(t)$ recursively

$$Q^{(0)}(t) = e^{(t-t_i)A(x_{t_i})} Q^{(0)}(t) \tag{18}$$

for $t_i < t_{i+1}$.

5.2 The diffusion limit

Since $\alpha^n_{(i,j)} = \alpha^{(0)}_{i,j}$ for all $n \geq 0$ and $(i,j) \in I$ we can verify that by defining $\alpha^{(1)}_{(i,j)} = 0$ for all $(i,j) \in I$ the conditions in theorem 3 hold. Therefore assuming

$$\lim_{n\to\infty} \sqrt{n} \left[\frac{Q^n(0)}{n} - Q^{(0)}(0) \right] =^d Q^{(1)}(0) \tag{19}$$

implies

$$\lim_{n\to\infty} \sqrt{n} \left[\frac{Q^n(t)}{n} - Q^{(0)}(t) \right] =^d Q^{(1)}(t) \tag{20}$$

where $Q^{(1)}(t)$ satisfies a SDE which may be expressed as the following integral equation

$$Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t A_t Q^{(1)}(t) + \sum_{l=1}^{k} \int_0^t \left(Q_l^{(0)}(t)\right)^{1/2} B_l dW_t^{(l)}$$

where W^l is a k-dimensional vector of independent standard Brownian motions for $l = 1, ..., k$. The matrices B_l have components

$$(B_l(t))_{ij} = \begin{cases} -\mu_{(i,j)}^{1/2} & \text{if } i = j \\ \mu_{(i,j)}^{1/2} & \text{if } l = i \neq j \\ 0 & \text{otherwise} \end{cases}$$

It is always possible rewrite (21) as

$$Q^{(1)}(t) = Q^{(1)}(0) + \int_0^t A_t Q^{(1)}(t) + \int_0^t B(s) d\hat{W}_t^{(l)}$$

where $\hat{W}_t$ is a k-dimensional vector of independent standard Brownian motions. In the case $k = 2$ the SDE is

$$\begin{aligned} dQ_1^{(1)}(t) &= -Q_1^{(1)}(t)(\mu_1(t) + \mu_{12}(t))dt + Q_2^{(1)}(t)\mu_{12}(t)dt \\ &\quad -(Q_1^{(0)}(t))^{1/2}(\mu_1^{1/2}(t)dW_{1,t}^{(1)} + \mu_{12}^{1/2}(t)dW_{2,t}^{(1)}) + (Q_2^{(0)}(t)\mu_{21}(t))^{1/2}dW_{1,t}^{(2)} \\ dQ_2^{(1)}(t) &= -Q_2^{(1)}(t)(\mu_2(t) + \mu_{21}(t))dt + Q_1^{(1)}(t)\mu_{21}(t)dt \\ &\quad -(Q_2^{(0)}(t))^{1/2}(\mu_2^{1/2}(t)dW_{2,t}^{(2)} + \mu_{21}^{1/2}(t)dW_{1,t}^{(2)}) + (Q_1^{(0)}(t)\mu_{12}(t))^{1/2}dW_{(2,t}^{(1)} \end{aligned}$$

That is equivalent to the the following SDE system

$$dQ^{(1)}(t) = A_t Q^{(1)} dt + B(t) d\hat{\mathbf{W}}_t$$

where

$$B(t) = \begin{pmatrix} \sigma_1(t) & 0 \\ \rho(t)\sigma_2(t) & \sqrt{1-\rho^2(t)}\sigma_2(t) \end{pmatrix}$$

$$\begin{aligned} \sigma_1^2(t) &= Q_1^{(0)}(t)(\mu_1(t) + \mu_{12}(t)) + Q_2^{(0)}(t)\mu_{21}(t) \\ \sigma_2^2(t) &= Q_2^{(0)}(t)(\mu_2(t) + \mu_{21}(t)) + Q_1^{(0)}(t)\mu_{12}(t) \\ \rho(t)\sigma_1(t)\sigma_2(t) &= -Q_1^{(0)}(t)\mu_{12}(t) - Q_2^{(0)}(t)\mu_{21}(t) \end{aligned}$$

and

$$\hat{\mathbf{W}}_t = (\hat{W}_t^{(1)}, \hat{W}_t^{(2)})^t$$

is a bi dimensional standard Brownian motion with independent components.

Assuming $A_t = A$ time independent, the solution of the SDE is given by

$$Q^{(1)}(t) = e^{tA}Q^{(1)}(0) + \int_0^t e^{(t-s)A}B(s)d\hat{\mathbf{W}}_s \tag{21}$$

with $Q^{(1)}(0) = \mathbf{0}$. The process is a stable Gaussian system with covariance matrix given by the integral

$$\mathrm{Cov}[Q^{(1)}(t), Q^{(1)}(t)] = \int_0^t e^{(t-s)A}B(s)B(s)^{tr}(e^{(t-s)A})^{tr}ds \tag{22}$$

that can be calculated numerically by solving a matrix ordinary differential equation (the Lyapunov equation).

In the general case of X_t taking values in $\mathcal{X}$ the process is defined similarly to the case of the fluid limit, this is

$$Q^{(1)}(t) = e^{(t-t_i)A_{t_i}}Q^{(1)}(t_i) + \int_{t_i}^t e^{(t-s)A_{t_i}}B(s)d\hat{W}_t \tag{23}$$

for $t_i < t < t_{i+1}$ where t_i is the time of the i-th jump of X_t.

6 Some numerical results

We assume a two state (two credit rates) system and 20, 50 and 100 elements. We assume a two state external random environment where jumps occur according to a standard Poisson process (rate 1). By sampling 1000 times the random environment we construct both the diffusion approximation and the exact distribution. The latter is obtained by integrating the Kolmogorov forward equation associated to the process using Runge-Kutta. The parameters are shown in the table.

	$X(t) = 0$	$X(t) = 1$
μ_1	0.1	0.2
μ_2	0.2	0.4
μ_{12}	0.3	0.3
μ_{21}	0.2	0.2

The exact and approximated distributions for the case of 20 elements and initial distribution 50/50 is shown in figure 5. We can observe that the fitting of the marginal distributions is outstanding despite the relatively small number of elements.

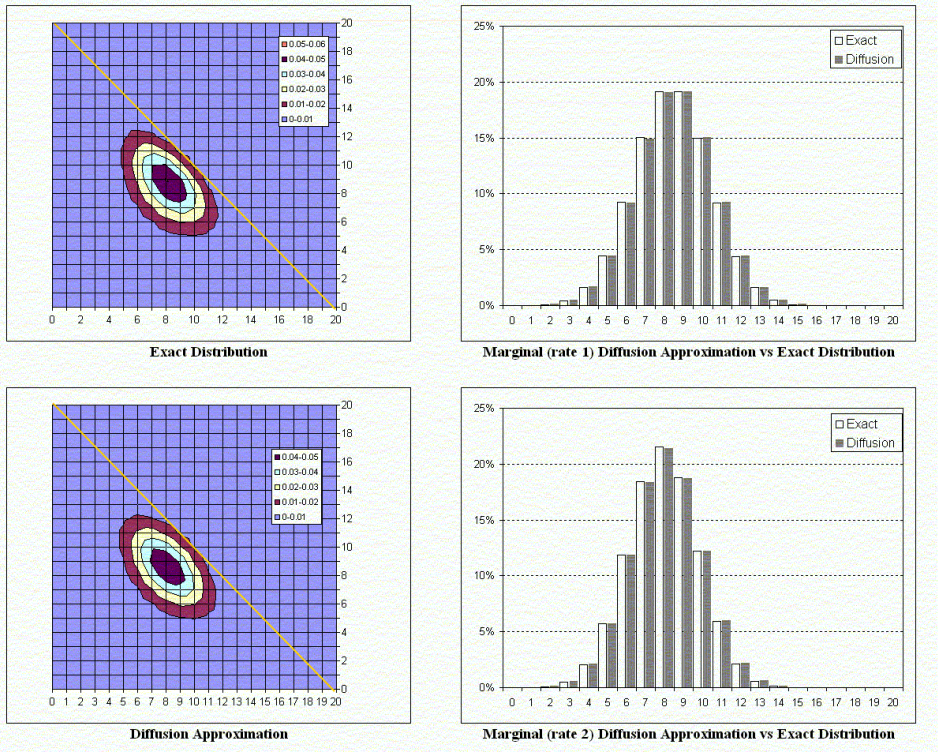


Figure 5. Exact and approximated distributions with 20 elements

We consider also the approximation to the number of survivors for different number of elements and initial distribution. The results are shown in figure 6. We can observe the effect of the random process $X(t)$ in the skewed distribution obtained.

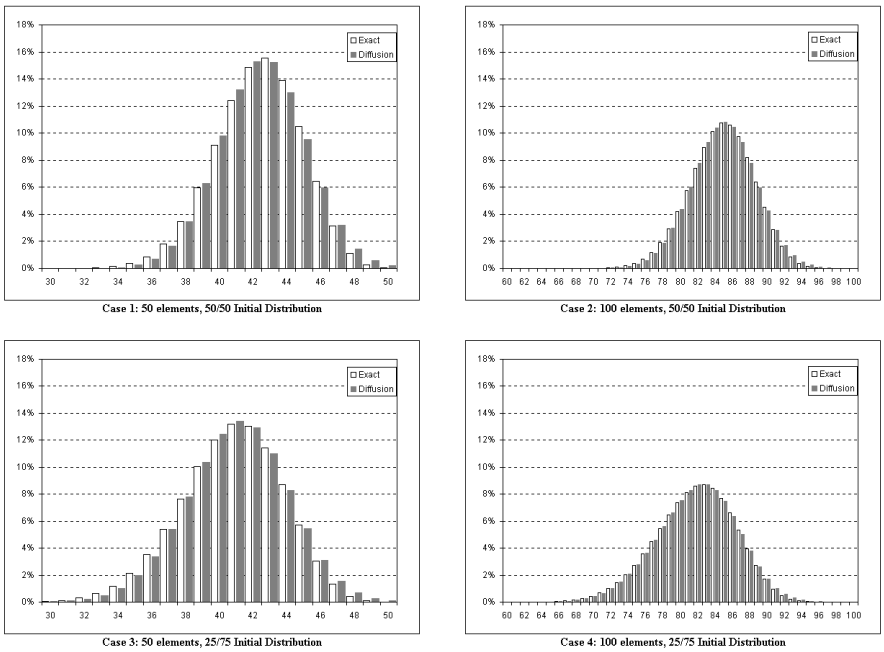


Figure 6. Approximation to the number of survivors

7 Concluding Remarks

In this paper we present a method for credit risk modelling that seems effective in predicting default performance with low computational cost. We present encouraging results that suggest the convenience of using limit approximations to the distribution of the rating distribution in a credit risk portfolio.

Further research involve investigating pricing applications. By now, we have no methodology to calibrate to the market information. Particularly, we are interested in the calibration to iTraxx tranche quotes as a method to estimate implied basket correlation.

Other applications may include the use of empirical change-of-rating data for risk management purposes.

A Appendix

We present the proof of the main theorems shown in this paper. Most of the proofs are taken from Choudhury, Mandelbaum et al. (1998). First we show the existence of the sequence of processes Q^η and the tightness property. In order to do that, lemma A.1 constructs a bound for the sequence.

Lemma A.1: *The process defined as solution of the equation*

$$Z^{\eta}(t) = X^{\eta}\left(\int_0^t \beta(s) Z^{\eta}(s) ds\right) \tag{24}$$

with

$$X^{\eta}(t) = X(0) + \sum_{i \in I} \frac{1}{\eta} A_i\left(\eta t\right) \mathbf{v}_i \tag{25}$$

is a pure jump process (no explosions) and the compensator is

$$\beta(s) \sum_{i \in I} |\mathbf{v}_i|$$

Proof. It can be seen that for any $t > 0$

$$E(X^{\eta}(t)) \leq E(X^{\eta}(0)) + \sum_{i \in I} \frac{1}{\eta} (\eta t) |\mathbf{v}_i| < \infty \quad \text{a.s.}$$

Now, if (24) has a solution we can define $\tau(t)$ as

$$\tau(t) = \int_0^t \beta(s) Z(s) ds = \int_0^t \beta(s) X(\tau(s)) ds$$

We need to prove we can define $\tau(t)$ uniquely and that $\tau(t) < \infty$ if $t < \infty$ a.s. Let us consider the one to one mapping $\tau(t)$ from t such that

$$\int_0^t \beta(s) ds = \int_0^{\tau(t)} \frac{1}{X(u)} du \tag{26}$$

and since by definition $X(u) > 0$ and $\beta(s) > 0$ then the mapping is uniquely defined, moreover since $X(u)$ has no explosions $\tau(t) < \infty$ as long as $t < \infty$ a.s. Finally, $\tau(t)$ is a stopping time with respect to the natural filtration of $X(t)$ from the definition of the mapping.

Now by differentiating both sides in (26)

$$\beta(t) = \frac{1}{X(\tau(t))} \frac{d\tau}{dt}$$

and then we define

$$Z(t) = \frac{\tau'(t)}{\beta(t)}$$

that satisfies

$$\tau(t) = \int_0^t \beta(s) Z(s) ds$$

and has the properties we required.

Now, we know that

$$x(t) - t\sum_{i\in I} |\mathbf{v}_i|$$

is a martingale, since $\tau(t)$ is a stopping time

$$M(t) = Z(t) - \tau(t)\sum_{i\in I} |\mathbf{v}_i|$$

is a martingale with respect to the natural filtration of $M(t)$ and then the compensator property is proved. ■

Now we show the existence and properties of the sequences Q^η and from this result is automatic theorem 1.

Theorem A.2: *For a set of indices I let $\{A_i|i \in I\}$ be a collection of standard Poisson processes in the space (Ω, F, P) and $X(s)$ a environment process as specified above. Let $\{\mathbf{v}_i|i \in I\}$ a collection of vectors in R^k.*

Consider the class of processes $Q^\eta(t)$, $\eta > 0$ of the form

$$Q^\eta(t) = Q^\eta(0) + \sum_{i\in I} A_i \left(\int_0^s \alpha_i^\eta \left(s, \frac{1}{\eta} Q^\eta(s), X(s) \right) ds \right) \mathbf{v}_i \tag{27}$$

where $\{\alpha_i^\eta|\eta > 0, i \in I\}$ is a family of measurable functions such that

$$|\alpha_i^\eta(s, y, x)| \leq \eta C(1 + |y|) \tag{28}$$

and $\{Q^\eta(0)|\eta > 0\}$ a family of rv's independent of the collection $\{A_i\}$.

For each $\eta > 0$, the process Q^η is uniquely defined (solution exists and it is unique) for almost ω in Ω.

Moreover, if

$$\overline{\lim_{\eta\to\infty}} \frac{|Q^\eta(0)|}{\eta} < \infty \quad \text{a.s.}$$

then

$$\overline{\lim_{\eta\to\infty}} \frac{|Q^\eta(t)|}{\eta} < \infty \quad \text{a.s.}$$

and therefore the sequence is tight.

PROOF: First we prove that $Q^\eta(t)$ can be defined (unique solution exists) for all $\omega_1 \in \Omega_1$. We use and induction argument to construct Q^η. For $\eta > 0$ define $Q_0^\eta(t) = Q^\eta(0)$, and

$$Q_n^\eta(t) = Q^\eta(0) + \sum_{i\in I} 1/\eta A_i \left(\int_0^{t\wedge T_n} \alpha_i^\eta \left(s, 1/\eta Q_{n-1}^\eta(s), X(s) \right) ds \right) \mathbf{v}_i \tag{29}$$

where

$$T_n = \inf\left\{ s : \sum_{i\in I} A_i \left(\int_0^t \alpha_i(t, 1/\eta Q^\eta_{n-1}(s), X(s))ds \right) = n \right\} \tag{30}$$

where $T_n = \infty$ if the set is empty. By construction $Q^\eta_0(t) = Q^\eta_1(t)$ for $0 \le t < T_1$. Using the induction hypothesis $Q_n(t) = Q_{n-1}(t)$ for $t < T_n$ implies

$$\int_0^t \alpha^\eta_i(s, 1/\eta Q^\eta_n(s), X(s))ds = \int_0^t \alpha^\eta_i(s, 1/\eta Q^\eta_{n-1}(s), X(s))ds$$

for $t < T_n$ and $i \in I$. By continuity of the integral, the same equality holds for $t = T_n$. So $Q^\eta_n(t) = Q^\eta_{n+1}(t)$ for $t \le T_n$. Now by the properties of the Poisson process (pure jump process) $Q^\eta_{n+1}(t) = Q^\eta_n(t) = Q^\eta_n(T_n)$ for $T_n \le t < T_{n+1}$. Then $Q^\eta_n(t) = Q^\eta_{n-1}(t)$ for $0 \le t < T_n$ and $Q\eta(t) = Q^\eta_n(t)$ for $T_n \le t < T_{n+1}$ and $Q^\eta(t)$ is defined in unique way.

Now consider the process defined by the SDE

$$Z(t) = 1 + \frac{1}{\eta}|Q^\eta(0)| + \sum \frac{1}{\eta} A_i \left(\eta C \int_0^t Z(s)ds \right) |\mathbf{v}_i|$$

such process is well defined (Lemma A.1). Notice that if

$$1 + \frac{1}{\eta}|Q^\eta_{n-1}(s)| < Z^\eta(s)$$

for $t < T_n$ almost surely, then

$$\begin{aligned}
1 + \frac{1}{\eta}|Q^\eta_n(t)| &\le 1 + \frac{1}{\eta}|Q^\eta(0)| + \sum_{i\in I} \frac{1}{\eta} A_i \left(\int_0^t \alpha^\eta_i(s, 1/\eta Q^\eta_{n-1}(s), X(s))ds \right) |\mathbf{v}_i| \\
&\le 1 + \frac{1}{\eta}|Q^\eta(0)| + \sum_{i\in I} \frac{1}{\eta} A_i \left(\eta C \int_0^t (1 + 1/\eta |Q_{n-1}(s)|)ds \right) |\mathbf{v}_i| \\
&\le 1 + \frac{1}{\eta}|Q^\eta(0)| + \sum_{i\in I} \frac{1}{\eta} A_i \left(\eta C \int_0^t Z^\eta(s))ds \right) |\mathbf{v}_i|
\end{aligned}$$

almost surely, therefore we can conclude that

$$\sup_{0\le s\le t} 1 + |Q^\eta_n(t)| < Z^\eta(s) \tag{31}$$

for $t > 0$ a.s.

Notice that for any $i \in I$ and $n \ge 0$

$$\begin{aligned}
A_i \left(\int_0^t \alpha^\eta_i \left(s, 1/\eta Q^\eta_n(s), X(s)ds\right) \right) &\le A_i \left(C \int_0^t 1 + 1/\eta |Q^\eta(t)|ds \right) \\
&\le A_i \left(C \int_0^t Z^\eta(t)ds \right)
\end{aligned}$$

then

$$\sum_{i\in I} A_i\left(\int_0^t \alpha_i^\eta\left(s, 1/\eta Q_n^\eta(s), X(s)ds\right)\right) \leq \sum_{i\in I} A_i\left(C\int_0^t Z^\eta(t)ds\right)$$

and since $\{A_i\}$ is a collection of independent standard Poisson process, the sum can be represented as a Poisson process A^*, this is

$$\sum_{i\in I} A_i\left(C\int_0^t Z^\eta(t)ds\right) = A^*\left(\eta C\eta\int_0^t Z^\eta(t)\right)$$

that is a well behaved pure jump process, therefore we can define for τ_n the time of the n-th jump in A^* a sequence $t(\tau_n)$ such that

$$\tau_n = \eta C\int_0^{t(\tau_n)} Z^\eta(t)ds$$

Clearly $t(\tau)\to\infty$ a.s. as $\eta\to\infty$. Additionally $t(\tau_n)\leq T_n$, otherwise the inequality is broken, therefore $T_n\to\infty$ a.s.

Finally, assume

$$\overline{\lim_{\eta\to\infty}}\frac{|Q^\eta(0)|}{\eta} \leq \infty \quad \text{a.s.}$$

then recall (31) , this is

$$\sup_{0\leq s\leq t} 1 + |Q_n^\eta(t)| < Z^\eta(s)$$

and lemma 1 that implies

$$Z^\eta(t)\exp\left\{-Ct\sum_{i\in I}|\mathbf{v}_i|\right\}$$

is a martingale, therefore

$$E\{Z^\eta(t)\} = \left(1+\frac{1}{\eta}|Q^\eta(0)|\right)\exp\left\{Ct\sum_{i\in I}|\mathbf{v}_i|\right\}$$

and using the Tchebychev inequality se can prove the sequence $Z^\eta(t)$ is tight. Therefore $\overline{\lim} Z^\eta(t)$ exists and is uniformly integrable, then

$$\overline{\lim_{\eta\to\infty}} Z^\eta(t) < \infty \quad \text{a.s.}$$

and consequently

$$\overline{\lim_{\eta\to\infty}} Q^\eta(t) < \infty \quad \text{a.s.} \qquad \square$$

The strong approximation result is derived from a theorem appeared in Ethier and Kurtz (1986) and previously in Kurtz (1978). We present the enunciation as a lemma without proof.

Lemma A.3:[Kurtz (1986)] *A standard Poisson process* $\{A(t)|t \geq 0\}$ *can be realized in the same space as a standard Brownian motion* $\{B(t)|t \geq 0\}$ *in such way that the positive r.v.* X *given by*

$$X = \frac{|A(t) - t - B(t)|}{ln(2 \vee t} < \infty \tag{32}$$

has a finite moment generating function in a neighborhood of the origin, in particular has finite mean.

With this result we show the convergence of the series $1/\eta Q^\eta$ to afterwards show the situations where the limit is indeed $Q^{(0)}$.

Lemma A.4: *If* $\{\alpha_i^\eta | \eta > 0, i \in I\}$ *is a family of measurable functions such that (28) holds and* $\{A_i\}$, $X(t)$, $\{Q^\eta(0)\}$ *and* $\{\mathbf{v}_i\}$ *are as defined above, then*

$$\limsup \frac{1}{\ln \eta} \left| Q^\eta(t) - Q^\eta(0) - \left[\int_0^t \alpha^\eta \left(s, 1/\eta Q^\eta(s), X(s)\right) ds \right.\right.$$
$$\left.\left. + \sum_{i \in I} B_i \left(\int_0^t \alpha_i^\eta \left(s, 1/\eta Q(s), X(s)\right) ds \right) |\mathbf{v}_i| \right] \right| = C_1 < \infty \tag{33}$$

as $\eta \to \infty$ *for almost all* ω *in* Ω.

Proof: The term inside the norm in the left hand side of equation (33) is equal to

$$\left| \sum_{i \in I} A_i \left(\int_0^t \alpha_i^\eta \left(s, 1/\eta Q(s), X(s)\right) ds \right) |\mathbf{v}_i| - \int_0^t \alpha^\eta \left(s, 1/\eta Q^\eta(s), X(s)\right) ds \right.$$
$$\left. - \sum_{i \in I} B_i \left(\int_0^t \alpha_i^\eta \left(s, 1/\eta Q(s), X(s)\right) ds \right) |\mathbf{v}_i| \right|$$

and applying lemma A.3 we can bound the term by

$$\leq \sum_{i \in I} X_i \ln \left(2 \vee \int_0^t \alpha_i^\eta \left(s, 1/\eta Q^\eta(s), X(s)\right) ds \right) |\mathbf{v}_i|$$
$$\leq \sum_{i \in I} X_i \ln \left(2 \vee \eta C \int_0^t 1 + \frac{1}{\eta} |Q^\eta(s)| ds \right) ln \left(2 \vee \eta C \int_0^t Z^\eta(t) ds \right) |\mathbf{v}_i|$$
$$\leq \sum_{i \in I} X_i |\mathbf{v}_i|$$

where $Z^\eta(t)$ is as in the proof of theorem A.2. By lemma A.3

$$E\left[\sum X_i |\mathbf{v}_i| \right] < \infty$$

and

$$\frac{\ln\left(2 \vee \eta C \int_0^t Z^\eta(t)ds\right)}{\ln(\eta)} \to C_2 < \infty$$

as $\eta \to \infty$ and the lemma is proved $\square$

Once the strong convergence is guaranteed, we derive the strong limit when the default rates are Lipschitz bounded, for this we require a generalisation of the Gronwall inequality we present without proof.

Lemma A.5: *Let x, y and z be measurable, non-negative functions on the reals. If y is bounded and z is integrable on $[0,T]$ and for all $0 \leq t \leq T$*

$$x(t) \leq z(t) + \int_0^t s(x)y(s)ds$$

then

$$x(t) \leq z(t) + \int_0^t z(s)y(s)\exp\left[\int_0^t y(r)dr\right]$$

and

$$\sup_{0\leq t\leq T} x(t) \leq \sup_{0\leq t\leq T} z(t)\exp\left[\int_0^T y(s)ds\right]$$

Theorem A.6: (Theorem 2) *If $\{\alpha_i^\eta | i \in I, \eta > 0\}$ are Lipschitz bounded and*

$$\lim_{\eta\to\infty} \sum_{i=0}^N \int_0^t \left\| \frac{\alpha_{i,x}^\eta(s,\cdot)}{\eta} - \alpha_{i,x}^{(0)}(s,\cdot) \right\| ds = 0 \tag{34}$$

For P^1 a.e. $\omega_1 \in \Omega^1$ the process $Q^\eta(t)$ is that

$$\frac{Q^\eta(0)}{\eta} \to Q^{(0)}(0) \text{ as } \eta \to \infty \tag{35}$$

then

$$\frac{Q^\eta(t)}{\eta} \to Q^{(0)}(t) \text{ as } \eta \to \infty \tag{36}$$

a.s. in P^1, where $Q^{(0)}(t)$ defined in R^K is the solution to the equation

$$Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \alpha^{(0)}(s, Q^{(0)}(s), X(s))ds \tag{37}$$

where

$$\alpha^{(0)}(t,\cdot,x) = \sum_{i=1}^N \alpha_i^{(0)}(t,\cdot,x)\mathbf{v}_i \tag{38}$$

and the integral equation is deterministic conditional on the random environment process $X(t)$.

PROOF: We need to prove that:

1. The function $\alpha^{(0)}$ is well defined, this is, the function exists and it is Lipschitz.

2. For all ω in Ω

$$\lim_{\eta\to\infty}\sup_{0\le s\le t}\left|\frac{1}{\eta}Q^{\eta}(s)-Q^{(0)}(s)\right|=0 \tag{39}$$

almost surely under P^{ω_1}.

First, notice that

$$\begin{aligned}\int_0^t\|\alpha^{(0)}\|ds &\le \int_0^t\|\alpha^{\eta}(s,y,X(s))\| \\ &\quad+\int_0^t\left\|\frac{1}{\eta}\alpha^{\eta}(s,y,X(s))-\alpha^{(0)}(s,y,X(s))\right\|ds \\ &\le \sup|\mathbf{v}_i|\sum_{i\in I}\left[\int_0^t\beta(s)ds\right. \\ &\quad\left.+\int_0^t\left\|\frac{1}{\eta}\alpha_i^{\eta}(s,y,X(s))-\alpha_i^{(0)}(s,y,X(s))\right\|ds\right]<\infty\end{aligned}$$

and since I is finite it is clear that

$$\lim_{\eta\to\infty}\int_0^t\left\|\frac{1}{\eta}\alpha^{\eta}(s,y,X(s))-\alpha^{(0)}(s,y,X(s))\right\|ds=0$$

otherwise it can be derived form the inequality

$$\begin{aligned}&\int_0^t\left\|\frac{1}{\eta}\alpha^{\eta}(s,y,X(s))-\alpha^{(0)}(s,y,X(s))\right\|ds \\ &\qquad\le\sup_{0\le s\le t}\sum_{i\in I}\int_0^t\left\|\frac{1}{\eta}\alpha_i^{\eta}(s,y,X(s))-\alpha_i^{(0)}(s,y,X(s)\right\|\end{aligned}$$

where the right hand side converges to zero as $\eta\to 0$.

Now as a consequence of lemma A.4 we have

$$\begin{aligned}\frac{Q^{\eta}(t)}{\eta}-Q^{(0)}(t) &= \int_0^t\frac{1}{\eta}\alpha^{\eta}(s,\frac{1}{\eta}Q^{\eta}(s),X(s))-\alpha^{(0)}(s,Q^{(0)},X(s))ds \\ &\quad+\sum_{i\in I}\frac{1}{\eta}B_i\left(\alpha_i^{\eta}\left(s,1/\eta Q^{\eta}(s),X(s)\right)ds\right)\mathbf{v}_i+\frac{1}{\eta}O(\ln\eta)\end{aligned}$$

where $O(\ln\eta)$ is a function such that $O(\ln\eta)/\ln(\eta)\to C<\infty$ as $\eta\to\infty$.

Now

$$\sup_{0\le s\le t}\quad \left|\frac{Q^\eta(s)}{\eta}-Q^{(0)}(s)\right|\le$$
$$\int_0^t\left|\frac{1}{\eta}\alpha^\eta\left(s,1/\eta Q^\eta(s),X(s)\right)-\alpha^{(0)}\left(s,1/\eta Q^\eta(s),X(s)\right)\right|ds$$
$$+\int_0^t\left|\alpha^{(0)}\left(s,1/\eta Q^\eta(s),X(s)\right)-\alpha^{(0)}\left(s,Q^0(s),X(s)\right)\right|ds$$
$$+\sum_{i\in I}\sup_{0\le s\le t}\frac{1}{\eta}\left|\left|B_i\left(\int_0^s\alpha_i^\eta\left(r,1/\eta Q^\eta(u),X(u)\right)du\right)\right|\mathbf{v}_i\right|+\frac{1}{\eta}O(\ln\eta)$$

By lemma A.3 and the Lipschitz property of α^η

$$\le\int_0^t\left\|\frac{\alpha^\eta(s,y,X(s))}{\eta}-\alpha^{(0)}(s,y,X(s))\right\|Z^\eta(t)$$
$$+\int_0^t\|\alpha^\eta(s,y,X(s))\|\sup_{0\le r\le s}\left|\frac{Q^\eta(r)}{\eta}-Q^{(0)}(r)\right|ds$$
$$+\sum_{i\in I}\sup_{0\le s\le t}\frac{1}{\eta}\left|B_i\left(C\int_0^t Z^\eta(s)ds\right)\right|\mathbf{v}_i|+\frac{1}{\eta}O(\ln\eta)$$

and by lemma A.5 (generalisation of Gronwall inequality)

$$\sup_{0\le s\le t}\quad\left|\frac{Q^\eta(s)}{\eta}-Q^{(0)}(s)\right|\le\exp\left(\int_0^t\|\alpha^{(0)}(s,y,X(s))\|ds\right)$$
$$\left(\int_0^t\left\|\frac{\alpha^\eta(s,y,X(s))}{\eta}-\alpha^{(0)}(s,y,X(s))\right\|\right)$$
$$+\sum_{i\in I}\sup_{0\le s\le t}\frac{1}{\eta}\left|B_i(\eta CsZ^\eta(s))\right|\mathbf{v}_i|+\frac{1}{\eta}O(\ln\eta)$$

and since $\|\alpha^{(0)}\|\le\infty$ and $\int\|\alpha^\eta-\alpha^{(0)}\|\to 0$ as $\eta\to\infty$, we conclude

$$\lim_{\eta\to\infty}\sup_{0\le s\le t}\left|\frac{Q^\eta(s)}{\eta}-Q^{(0)}(s)\right|=0\qquad\square$$

Finally, we prove the weak convergene result. Before proving theorem 3 we need to prove that the sequence

$$\frac{1}{\eta}|Q^\eta(s)-\eta Q^{(0)}|$$

has a subsequence that is tight and then the weak convergence is possible.
Lemma A.7: *For all $t>0$ if $\{\alpha_i^\eta(s,y,x)|\eta>0,i\in I\}$ is a family of positive Lipschitz bounded functions as in theorem 2 assumptions, and*

$$\sum_{i\in I}\overline{\lim_{\eta\to\infty}}\sup_{0\le s\le t}\frac{1}{\sqrt{\eta}}\|\alpha^\eta(s,y,X(s))-\eta\alpha^{(0)}(s,y,X(s))\|ds<\infty\qquad(40)$$

P^1 *a.s. then the sequence*

$$\frac{1}{\eta}|Q^\eta(s) - \eta Q^{(0)}(s)|$$

is such that

$$P\left\{\frac{1}{\eta}|Q^\eta(s) - \eta Q^{(0)}(s)| < \infty\right\} = 1$$

PROOF: The proof follows the same arguments as in the proof of theorem 2.

From condition (40)

$$\sum_{i\in I} \overline{\lim_{\eta\to\infty}} \sup_{0\le s\le t} \frac{1}{\eta}\|\alpha^\eta(s,y,X(s)) - \eta\alpha^{(0)}(s,y,X(s))\|ds = 0$$

and then condition (34) holds and theorem 2 is valid. Then $\|\alpha^{(0)}\|$ is a Lipschitz map. Moreover, from the proof of theorem 2 we recall

$$\int_0^t \left\|\frac{1}{\eta}\alpha^\eta(s,y,X(s)) - \alpha^{(0)}(s,y,X(s))\right\| ds$$
$$\le \sup_{0\le s\le t} \sum_{i\in I} \int_0^t \left\|\frac{1}{\eta}\alpha_i^\eta(s,y,X(s)) - \alpha_i^{(0)}(s,y,X(s)\right\|$$

and multiplying by $\sqrt{\eta}$

$$\int_0^t \frac{1}{\sqrt{\eta}}\|\alpha^\eta(s,y,X(s)) - \eta\alpha^{(0)}(s,y,X(s))\| < \infty$$

P a.s. for all $t > 0$

By Lemma A.4

$$\begin{aligned} Q^\eta(t) &= Q^\eta(0) + \int_0^t \alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) ds \\ &\quad + \sum_{i\in I} B_i\left(\int_0^t \alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) ds\right)\mathbf{v}_i + O(\ln\eta) \\ &=^d Q^\eta(0) + \int_0^t \alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) ds \\ &\quad + \sum_{i\in I} \sqrt{\eta} B_i'\left(\frac{1}{\eta}\int_0^t \alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) ds\right)\mathbf{v}_i + O(\ln\eta) \end{aligned}$$

where $\{B_i'\}$ is a collection of independent Brownian motions different to the family $\{B_i\}$ that can be constructed in the same probability space by the self-similarity properties of the Bm's.

Then

$$\begin{aligned}\frac{1}{\sqrt{\eta}} \quad & \left[Q^\eta(t) - \eta Q^{(0)}(t)\right] =^d \\ & \frac{1}{\sqrt{\eta}} \int_0^t \alpha^\eta\left(s, 1/\eta Q(s), X(s)\right) - \eta\alpha^{(0)}(s, Q^{(0)}(s), X(s))ds \\ & + \sum_{i\in I} B_i'\left(\frac{1}{\eta}\int_0^t \alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) ds\right)\mathbf{v}_i + \frac{1}{\sqrt{\eta}}O(\ln\eta) \\ & = \frac{1}{\sqrt{\eta}}\Theta(t)\end{aligned}$$

We then work with $\Theta(t)$, it can be seen that

$$\begin{aligned}\sup_{0\le s\le t}\frac{1}{\sqrt{\eta}}\Theta(t) \le & \frac{1}{\sqrt{\eta}}\int_0^t \left|\alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) - \eta\alpha^{(0)}\left(s, 1/\eta Q^\eta(s), X(s)\right)\right| ds \\ & + \frac{1}{\sqrt{\eta}}\int_0^t \eta\left|\alpha^{(0)}\left(s, 1/\eta Q^\eta(s), X(s)\right) - \alpha^{(0)}\left(s, 1/\eta Q^{(0)}(s), X(s)\right)\right| ds \\ & + \sum_{i\in I}\sup_{0\le s\le t}\left|B_i'\left(\frac{1}{\eta}\int_0^s \alpha^\eta\left(u, 1/\eta Q^\eta(u), X(u)\right) du\right)\right| |\mathbf{v}_i| + \frac{1}{\sqrt{\eta}}O(\ln\eta)\end{aligned}$$

By the Lipschitz continuity of $\alpha^{(0)}$ and α^η

$$\begin{aligned}\le & \int_0^t \frac{1}{\sqrt{\eta}}\|\alpha^\eta(s, y, X(s)) - \eta\alpha^{(0)}(s, y, X(s))\| ds Z^\eta(t) \\ & + \int_0^t \|\alpha^{(0)}(s, y, X(s)\|\frac{1}{\eta}\sup_{0\le r\le s}\left|Q^\eta(r) - \eta Q^{(0)}(r)\right| ds \\ & + \sum_{i\in I}\sup_{0\le s\le t}\left|B_i'(sCZ^\eta(s))\right| |\mathbf{v}_i| + \frac{1}{\sqrt{\eta}}O(\ln\eta)\end{aligned}$$

and by lemma A.5

$$\begin{aligned}\sup_{0\le s\le t}\frac{1}{\sqrt{\eta}} \quad \le \quad & \exp\left[\int_0^t \|\alpha^{(0)}(s, y, X(s))\| ds\right] \\ & \left(\int_0^t \frac{1}{\sqrt{\eta}}\|\alpha^\eta(s, y, X(s)) - \eta\alpha^{(0)}(s, y, X(s))\| ds Z^\eta(t)\right. \\ & \left. + \sum_{i\in I}\sup_{0\le s\le t}\left|B_i'(sCZ^\eta(s))\right| |\mathbf{v}_i| + \frac{1}{\sqrt{\eta}}O(\ln\eta)\right)\end{aligned}$$

By taking limits in both sides

$$\overline{\lim_{\eta\to\infty}}\frac{1}{\sqrt{\eta}}Q^\eta(t) \le \exp\left[\int_0^t \|\alpha^{(0)}(s, y, X(s)\| ds\right]\sum_{i\in I}\sup_{0\le s\le t}\left|B_i'(sCZ^\eta(s))\right| < \infty$$

and then

$$P\left[\frac{1}{\eta}|Q^{\eta}(t) - \eta Q^{(0)}(t)| < \infty\right] = 1 \qquad \square$$

Finally, we show the weak approximation $Q^{(1)}$ to the sequence.

Theorem A.8: (Theorem 5) *Assume*

$$\sum_{i\in I}\int_0^t \overline{\lim_{\eta\to\infty}}\sqrt{\eta}\left\|\frac{\alpha_i^{\eta}(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x)\right\| ds < \infty \tag{41}$$

and

$$\lim_{\eta\to\infty}\sum_{i\in I}\int_0^t \left\|\sqrt{\eta}\left[\frac{\alpha_i^{\eta}(s,\cdot,x)}{\eta} - \alpha_i^{(0)}(s,\cdot,x)\right] - \alpha_i^{(1)}(s,\cdot,x)\right\| ds = 0 \tag{42}$$

and for all $x \in \mathcal{X}$ and $i \in I$. Assume the function $\alpha^{(0)}(s,\cdot,x)$ has scalable Lipschitz derivative for any values $X(t)$ and $Q^{(0)}(t)$. For P^1 a.e. $\omega_1 \in \Omega^1$, If

$$\lim_{\eta\to\infty}\sqrt{\eta}\left[\frac{Q^{\eta}(0)}{\eta} - Q^{(0)}(0)\right] =^d Q^{(1)}(0)$$

w.r.t $P_Q^{\omega_1}$, then

$$\lim_{\eta\to\infty}\sqrt{\eta}\left[\frac{Q^{\eta}(t)}{\eta} - Q^{(0)}(t)\right] =^d Q^{(1)}(t) \tag{43}$$

w.r.t $P_Q^{\omega_1}$, where the process $Q^{(1)}(t)$ takes values in Ω^2 and it is the solution to the stochastic integral equation

$$\begin{aligned} Q^{(1)}(t) &= Q^{(1)}(0) + \int_0^t \Lambda\alpha_i^{(0)}(s, X_s, Q^{(0)}; Q^{(1)}(s)) + \alpha^{(1)}(s, Q^{(0)}(s), X_s)ds \\ &\quad + \sum_{i\in I} B_i\left(\int_0^t \alpha_i^{(0)}(s, Q^{(0)}(s), X_s)ds\right)\mathbf{v}_i \end{aligned} \tag{44}$$

where $\{B_i(t)\}$ is a collection of independent standard Brownian motions in $(\Omega^2, \mathcal{F}^2, P^{\omega_1})$.

PROOF: We need to prove the following:

1. The limit rates $\alpha^{(0)}$ and $\alpha^{(1)}$ are Lipschitz bounded and their Lipschitz norm locally integrable as functions of time.

2. That

$$\lim_{\eta\to\infty}\sup_{0\le s\le t}\left|\frac{Q^{\eta}(s) - \eta Q^{(0)}(s)}{\sqrt{\eta}} - Q^{(1)}(s)\right| = 0 \tag{45}$$

   for almost all $\omega \in \Omega$.

If

$$\sum_{i\in I}\int_0^t \overline{\lim_{\eta\to\infty}}\sqrt{\eta}\left\|\frac{\alpha_i^\eta(s,\cdot,x)}{\eta}-\alpha_i^{(0)}(s,\cdot,x)\right\|ds<\infty$$

then

$$\sum_{i\in I}\int_0^t \lim_{\eta\to\infty}\left\|\frac{\alpha_i^\eta(s,\cdot,x)}{\eta}-\alpha_i^{(0)}(s,\cdot,x)\right\|ds=0$$

and then condition (34) of theorem 2 holds and $\alpha^{(0)}$ is Lipschitz bounded and $\|\alpha^{(0)}\|$ is locally integrable.

By a similar argument to the used in the proof of theorem 2 from eq (42)

$$\int_0^t\left\|\alpha^{(1)}\right\|ds\leq\int_0^t\left\|\left[\frac{\alpha^\eta-\eta\alpha^{(0)}}{\sqrt{\eta}}\right]\right\|ds+\int_0^t\left\|\left[\frac{\alpha^\eta-\eta\alpha^{(0)}}{\sqrt{\eta}}\right]-\alpha^{(1)}\right\|ds$$
$$\int_0^t\left\|\left[\frac{\alpha^\eta-\eta\alpha^{(0)}}{\sqrt{\eta}}\right]\right\|ds+\sup_{i\in I}\mathbf{v}_i\sum_{i\in I}\int_0^t\left\|\left[\frac{\alpha_i^\eta-\eta\alpha_i^{(0)}}{\sqrt{\eta}}\right]-\alpha_i^{(1)}\right\|ds<\infty$$

and therefore $\|\alpha^{(1)}\|$ is locally integrable. Moreover, since

$$\int_0^t\left\|\left[\frac{\alpha^\eta-\eta\alpha^{(0)}}{\sqrt{\eta}}\right]-\alpha^{(1)}\right\|ds<\sup_{i\in I}\mathbf{v}_i\sum_{i\in I}\int_0^t\left\|\left[\frac{\alpha_i^\eta-\eta\alpha_i^{(0)}}{\sqrt{\eta}}\right]-\alpha_i^{(1)}\right\|ds$$

and since $\sup_{i\in I}\mathbf{v}_i<\infty$ then

$$\lim_{\eta\to\infty}\int_0^t\left\|\left[\frac{\alpha^\eta-\eta\alpha^{(0)}}{\sqrt{\eta}}\right]-\alpha^{(1)}\right\|ds=0$$

and the first point is proved.

Now we proof the second point, we bound the term in left hand side of (45) and show each term goes to zero as $\eta\to\infty$.

$$\begin{aligned}
&\frac{1}{\sqrt{\eta}}\left[Q^\eta(t)-Q^{(0)}(t)\right]\\
&=^d\int_0^t\frac{1}{\eta}\left(\alpha^\eta\left(s,1/\eta Q^\eta(s),X(s)\right)-\eta\alpha^{(0)}\left(s,Q^{(0)}(s),X(s)\right)\right)ds\\
&-\int_0^t\Lambda\alpha^{(0)}\left(s,Q^{(0)}(s),X(s);Q^{(1)}(s)\right)+\alpha^{(1)}(s,Q^{(0)}(s),X(s))ds\\
&+\sum_{i\in I}\left[B_i'\left(\frac{1}{\eta}\int_0^t\alpha^\eta\left(s,1/\eta Q^\eta(s),X(s)\right)ds\right)\right.\\
&\left.-B_i'\left(\int_0^t\alpha^{(0)}\left(s,Q^{(0)}(s),X(s)\right)ds\right)\right]\mathbf{v}_i\\
&+\frac{1}{\sqrt{\eta}}O(\ln\eta)
\end{aligned}$$

and

$$
\begin{aligned}
&=^d \int_0^t \frac{1}{\eta}\left(\alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) - \eta\alpha^{(0)}\left(s, 1/\eta Q^\eta(s), X(s)\right)\right)\\
&-\alpha^{(1)}\left(s, Q^{(0)}(s), X(s)\right) ds\\
&+\int_0^t \left(\eta\alpha^{(0)}\left(s, 1/\eta Q^\eta(s), X(s)\right) - \eta\alpha^{(0)}\left(s, Q^{(0)}(s), X(s)\right)\right)\\
&-\Lambda\alpha^{(0)}\left(s, Q^{(0)}(s), X(s); Q^{(1)}(s)\right) ds\\
&+\int_0^t \alpha^{(1)}\left(s, Q^{(0)}(s), X(s)\right) - \alpha^{(1)}(s, Q^{(0)}(s), X(s)) ds\\
&+\sum_{i\in I}\left[B_i'\left(\frac{1}{\eta}\int_0^t \alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) ds\right)\right.\\
&\left.-B_i'\left(\int_0^t \alpha^{(0)}\left(s, Q^{(0)}(s), X(s)\right) ds\right)\right]\mathbf{v}_i\\
&+\frac{1}{\sqrt{\eta}}O(\ln\eta)
\end{aligned}
$$

and again

$$
\begin{aligned}
&=^d \int_0^t \frac{1}{\eta}\left(\alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) - \eta\alpha^{(0)}\left(s, 1/\eta Q^\eta(s), X(s)\right)\right)\\
&-\alpha^{(1)}\left(s, Q^{(0)}(s), X(s)\right) ds && \text{(46a)}\\
&+\int_0^t \left(\eta\alpha^{(0)}\left(s, 1/\eta Q^\eta(s), X(s)\right) - \eta\alpha^{(0)}\left(s, Q^{(0)}(s), X(s)\right)\right.\\
&\left.-\Lambda\alpha^{(0)}\left(s, Q^{(0)}(s), X(s); 1/\eta Q^\eta(s) - Q^{(0)}(s)\right)\right) ds && \text{(46b)}\\
&\int_0^t \sqrt{\eta}\Lambda\alpha^{(0)}\left(s, Q^{(0)}(s), X(s); \frac{1}{\eta}Q^\eta(s) - Q^{(0)}(s)\right)\\
&-\Lambda\alpha^{(0)}\left(s, Q^{(0)}(s), X(s); Q^{(1)}(s)\right) ds && \text{(46c)}\\
&+\int_0^t \alpha^{(1)}\left(s, Q^{(0)}(s), X(s)\right) - \alpha^{(1)}(s, Q^{(0)}(s), X(s)) ds && \text{(46d)}\\
&+\sum_{i\in I}\left[B_i'\left(\frac{1}{\eta}\int_0^t \alpha^\eta\left(s, 1/\eta Q^\eta(s), X(s)\right) ds\right)\right.\\
&\left.-B_i'\left(\int_0^t \alpha^{(0)}\left(s, Q^{(0)}(s), X(s)\right) ds\right)\right]\mathbf{v}_i && \text{(46e)}\\
&+\frac{1}{\sqrt{\eta}}O(\ln\eta)
\end{aligned}
$$

Amazingly all terms tend to zero as $\eta \to 0$. Now here is why. Using the same arguments as in the proof of lemma A.4, the first term (46a) can be

bounded by the pure process sequence

$$\int_0^t \left\| \frac{\alpha^\eta(s,y,X(s)) - \eta\alpha^{(0)}(s,y,X(s))}{\sqrt{\eta}} - \alpha^{(1)}(s,y,X(s)) \right\| dsZ^\eta(t)$$

The second term (46b) can be rewritten as

$$\sqrt{\eta}\left|\frac{1}{\eta}Q^\eta(s) - Q^{(0)}(s)\right| \frac{1}{\frac{1}{\eta}Q^\eta(s) - Q^{(0)}(s)} \left\| \alpha^{(0)}(s, 1/\eta Q^\eta(s), X(s)) - \alpha^{(0)}(s, Q^{(0)}(s), X(s)) - \Lambda\alpha^{(0)}(s, Q^{(0)}, X(s); 1/\eta Q^\eta(s) - Q^{(0)}(s)) \right\|$$

The right hand (first line and the rest) may be analysed using the properties of the scalable Lipschitz derivative while the left term is always bounded by theorem 2.

For the third term we use the fact that

$$\|\Lambda f(x;y)\| \leq \|f\|$$

therefore the term (46c) can be bounded by

$$\int_0^t \|\alpha^{(0)}(s,y,X(s))\| \overline{\lim_{\eta\to\infty}} \sup_{0\leq u\leq s} \left| \frac{Q^\eta(s) - \eta Q^{(0)}(u)}{\sqrt{\eta}} - Q^{(1)}(u) \right| \tag{47}$$

The term (46d) is bounded by the Lipschitz condition

$$\|\alpha^{(1)}(s,y,X(s))\| ds \sup_{0\leq s\leq t} \left| \frac{1}{\eta}Q^\eta(s) - Q^{(0)}(s) \right|$$

and by theorem 2 the term vanishes as $\eta \to \infty$.

For the final term (46e) we recall the paths of a Brownian motion are continuous with probability 1. Therefore if

$$\lim_{\eta\to\infty} \sup_{0\leq s\leq t} \left| \int_0^s \alpha^\eta \left(u, \frac{1}{\eta}Q^\eta(u), X(u)\right) du - \int_0^s \alpha^{(0)} \left(u, Q^{(0)}(u), X(u)\right) du \right| = 0$$

holding in compact sets, the term (46e) will vanish as well. Notice that

$$\begin{aligned}
&\sup_{0\leq s\leq t} \left| \frac{1}{\eta}\int_0^s \alpha_i^\eta \left(u, \frac{1}{\eta}Q^\eta(u), X(u)\right) du - \int_0^s \alpha_i^{(0)} \left(u, Q^{(0)}(u), X(u)\right) \right| \\
&\quad \leq \int_0^t \left| \frac{1}{\eta}\alpha_i^\eta \left(s, 1/\eta Q^\eta(s), X(s)\right) - \alpha_i^{(0)} \left(s, 1/\eta Q^\eta(s), X(s)\right) \right| ds \\
&\qquad + \int_0^t \left| \alpha_i^{(0)} \left(s, 1/\eta Q^\eta(s), X(s)\right) - \alpha_i^{(0)} \left(s, Q^{(0)}(s), X(s)\right) \right| ds \\
&\quad \leq \int_0^t \left\| \frac{1}{\eta}\alpha^\eta(s,y,X(s)) - \alpha^{(0)}(s,y,X(s)) \right\| dsZ^\eta(t) + \\
&\qquad \int_0^t \|\alpha^{(0)}(s,y,X(s))\| ds \sup_{0\leq s\leq t} \left| \frac{1}{\eta}Q^\eta(s) - Q^{(0)}(s) \right|
\end{aligned}$$

that vanishes as in proof of theorem 2. Then

$$\lim_{\eta\to\infty}\sup_{0\le s\le t}\left|\sum_{i\in I}\left[B_i'\left(\frac{1}{\eta}\int_0^s \alpha_i^\eta\left(u,1/\eta Q^\eta(u),X(u)\right)du\right)\right.\right.$$
$$\left.\left.-B_i'\left(\int_0^s \alpha^{(0)}(u,Q^{(0)}(u),X(u))du\right)\right]\mathbf{v}_i\right|=0$$

Therefore in the limit the whole expression may be bounded by (47), this is

$$\overline{\lim_{\eta\to\infty}}\sup_{0\le s\le t}\left|\frac{Q^\eta(s)-\eta Q^{(0)}(s)}{\sqrt{\eta}}-Q^{(1)}(s)\right|$$
$$\le\int_0^t\|\alpha^{(0)}(s,y,X(s))\|\overline{\lim_{\eta\to\infty}}\sup_{0\le u\le s}\left|\frac{Q^\eta(s)-\eta Q^{(0)}(s)}{\sqrt{\eta}}-Q^{(1)}(s)\right|$$

and applying the lemma A.5,the first inequality with $z(t)=0$, we obtain

$$\overline{\lim_{\eta\to\infty}}\sup_{0\le s\le t}\left|\frac{Q^\eta(s)-\eta Q^{(0)}(s)}{\sqrt{\eta}}-Q^{(1)}(s)\right|=0$$

and the proof is complete. □

References

[1] Christensen, J., E. Hansen and D. Lando (2004). *Confidence Sets for Continuous-Time Rating Transition Probabilities*. Preprint.

[2] Choudhury, G., A. Mandelbaum, M. Reiman and W. Whitt (1998). *Fluid Diffusion Limits for Queues on Slowly Changing Environments.* Stochastic Models 13. 121-146.

[3] Chow, Y. and H. Teicher (1978). *Probability Theory: Independence, Interchangeability, Martingales*. Springer Verlag.

[4] Crowder, M., M. Davis and G. Giampieri (2003). *A Hidden Markov Model of Default Interaction*. Preprint.

[5] CSFB (1997). *CreditRisk+: A credit risk management framework*. Credit Suisse Financial Products, London.

[6] Dshalalow, J.H. (1995). *An Anthology of Classical Queueing Methods*. Advances in Queueing: Theory, Methods and Open Problems. CRC Press. Probability and Stochastic Series.

[7] Ethier, S. and T. Kurtz (1986). *Markov processes: characterization and convergence*. Prob. and Math. Stat. Series. John Wiley and Sons.

[8] Frey, R. and J. Backhaus (2003). *Interacting defaults and counterparty risk: A Markovian approach.* Proceedings of Dependence Modelling for Credit Portfolios. GRETA Associati. September.

[9] Frydman, H. and T. Schuermann (2004). *Credit Rating Dynamics and Markov Mixture Models.* Wharton Financial Institutions Center Working Paper No. 04-15.

[10] Giesecke, K. and S. Weber (2003). *Cyclical correlations, credit contagion and portfolio losses. Proceedings of Dependence modelling for credit portfolios.* GRETA Associati. September.

[11] Glynn, D. and W. Whitt (1988). *Ordinary CLT and WLLN versions of* $L = \lambda W$. Mathematics of Operations Research. Vol. 13. No. 4. November.

[12] Gupton, G., C. Finger and M. Bhatia (1997). *CreditMetrics - Technical Document.* J.P. Morgan & Co. Incorporated, New York.

[13] Hull, J. and A. White (2004). *Valuation of a CDO and an* n^{th} *to default CDS without MonteCarlo Simulation.* Working Paper. University of Toronto.

[14] Iglehart, D. and W. Whitt (1970). *Multiple channel queues in heavy traffic I.* Adv. Appl. Prob. 2. 150-177.

[15] Jarrow, R., D. Lando and S. Turnbull (1997). *A Markov Model for the Term Structure of Credit Risk Spreads.* Review of Financial Studies, Vol. 10, No. 2, 481-523.

[16] Kurtz, T. G. (1978). *Strong approximation theorems for densitiy dependent markov chains.* Stoch. Proc. Appl. Vol. 6. 223-240. 1978.

[17] Laurent, J.P. and J. Gregory (2003). *Basket Default Swaps, CDOs and Factor Copulas.* Working Paper. BNP Paribas.

[18] Lemoine, A.J. (1978). *Networks of Queues: A Survey of Weak Convergence Results.* Management Science. Vol. 24. No. 11. 1175-1193.

[19] Mandelbaum, A., W. Massey and M. Reiman (1998). *Strong Approximations for Markovian Service Networks.* Queueing Systems 30. 149-201.

[20] Tavares, P., T. Nguyen, A. Chapovsky and I. Vaysburd (2004). *A Composite Basket Model.* Working Paper. Merrill Lynch.

[21] Vasiceck, O. (1987). *Probability of loss in a loan portfolio.* Moody's KMV. August.