Default Correlation in the Structured Derivatives Markets

Alan White
Rotman School of Management
University of Toronto
## Synthetic CDO Quotes: CDX IG 5-Year (Basis Points)

<table>
<thead>
<tr>
<th></th>
<th>14-Dec-05</th>
<th>14-Dec-06</th>
<th>14-Dec-07</th>
<th>28-Feb-08</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>46</td>
<td>33</td>
<td>75</td>
<td>151</td>
</tr>
<tr>
<td>0%</td>
<td>3%</td>
<td>39%</td>
<td>24%</td>
<td>45%</td>
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<td>3%</td>
<td>7%</td>
<td>117</td>
<td>83</td>
<td>245</td>
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<tr>
<td>7%</td>
<td>10%</td>
<td>29</td>
<td>16</td>
<td>100</td>
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<tr>
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<td>15%</td>
<td>18</td>
<td>7</td>
<td>72</td>
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<td>15%</td>
<td>30%</td>
<td>10</td>
<td>2</td>
<td>49</td>
</tr>
<tr>
<td>30%</td>
<td>100%</td>
<td>2</td>
<td>0.2</td>
<td>14</td>
</tr>
</tbody>
</table>
**Approximate Hazard Rate**

Hazard rate ≈ Index Spread / (1 – Recovery)

Recovery assumed to be 40%

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<td><strong>Index</strong></td>
<td>46</td>
<td>33</td>
<td>75</td>
<td>151</td>
</tr>
<tr>
<td><strong>Hazard Rate</strong></td>
<td>0.77%</td>
<td><strong>0.54%</strong></td>
<td>1.25%</td>
<td>2.52%</td>
</tr>
</tbody>
</table>
Simple CDO Model

- Ignore discounting
- Tranche is touched within 5 years:
  - Probability = $p$
  - Payoff = $1$
- Tranche is not touched within 5 years:
  - Probability $(1 - p)$
  - Payment = $5 \times \text{spread}$
- Equilibrium $p = 5 \times \text{spread} / (1 + 5 \times \text{spread})$
- Tranche is touched when Proportion Defaulting $\geq \frac{\text{Attachment}}{(1 - R)} = \pi_D$
Probability, $p$, Tranche is Touched within 5 Years

<table>
<thead>
<tr>
<th>Attachment</th>
<th>$\pi_D$</th>
<th>14-Dec-05</th>
<th>14-Dec-06</th>
<th>14-Dec-07</th>
<th>28-Feb-08</th>
</tr>
</thead>
<tbody>
<tr>
<td>3%</td>
<td>5.0%</td>
<td>5.54%</td>
<td><strong>3.99%</strong></td>
<td>10.91%</td>
<td>22.33%</td>
</tr>
<tr>
<td>7%</td>
<td>11.7%</td>
<td>1.43%</td>
<td><strong>0.81%</strong></td>
<td>4.76%</td>
<td>12.89%</td>
</tr>
<tr>
<td>10%</td>
<td>16.7%</td>
<td>0.87%</td>
<td><strong>0.33%</strong></td>
<td>3.47%</td>
<td>10.15%</td>
</tr>
<tr>
<td>15%</td>
<td>25.0%</td>
<td>0.49%</td>
<td><strong>0.10%</strong></td>
<td>2.39%</td>
<td>7.62%</td>
</tr>
<tr>
<td>30%</td>
<td>50.0%</td>
<td>0.11%</td>
<td><strong>0.01%</strong></td>
<td>0.70%</td>
<td>2.53%</td>
</tr>
</tbody>
</table>
Independent Defaults

- Assume
  - Independent Defaults
  - All firms have same default probability, \( q \)

- Probability of \( n \) or fewer defaults in a portfolio of size \( N \) is \( B(n, N, q) \)

- Probability of \( n \) or more defaults in a portfolio of size \( N \) is \( 1 - B(n - 1, N, q) \)
Independent Default Probability, $q$, to Match Probability Tranche is Touched

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<th>14-Dec-07</th>
<th>28-Feb-08</th>
</tr>
</thead>
<tbody>
<tr>
<td>3%</td>
<td>5.0%</td>
<td>2.7%</td>
<td>2.5%</td>
<td>3.2%</td>
<td>3.9%</td>
</tr>
<tr>
<td>7%</td>
<td>11.7%</td>
<td>6.4%</td>
<td>6.0%</td>
<td>7.5%</td>
<td>8.7%</td>
</tr>
<tr>
<td>10%</td>
<td>16.7%</td>
<td>8.9%</td>
<td>8.9%</td>
<td>11.1%</td>
<td>12.6%</td>
</tr>
<tr>
<td>15%</td>
<td>25.0%</td>
<td>16.2%</td>
<td>14.7%</td>
<td>18.2%</td>
<td>20.0%</td>
</tr>
<tr>
<td>30%</td>
<td>50.0%</td>
<td>36.6%</td>
<td>33.8%</td>
<td>39.2%</td>
<td>41.3%</td>
</tr>
</tbody>
</table>
Top Down Models

- Model aggregate loss or number of defaults without reference to specific firm characteristics
- Work in this area by Schönbucher (2005), Sidenius et al (2005), Giesecke and Goldberg (2005), Longstaff and Rajan (2008)
Longstaff Model

- Portfolio loss driven by three Poisson processes, \( N_1, N_2, N_3 \)

\[
L_t = 1 - \exp\left( -\gamma_1 N_{1t} - \gamma_2 N_{2t} - \gamma_3 N_{3t} \right)
\]

- Average loss sizes are \( \gamma_1 = 0.4\% \), \( \gamma_2 = 6\% \), \( \gamma_3 = 35\% \)
- Intensities follow \( d\lambda_{it} = \sigma_i \sqrt{\lambda_{it}} dZ_{it} \)
- Average intensities are approximately \( \lambda_1 = 0.84 \), \( \lambda_2 = 0.024 \), \( \lambda_3 = 0.0013 \)
Reduced Form Models

- Model the hazard rate process for each name in the portfolio
- Work in this area by Duffie and Garleanu (2001) and many other authors
- Hazard rate processes are correlated jump-diffusion processes with both independent and common jumps
Reduced Form Models - Implementation

- For each name the hazard rate process is simulated from time zero to T
- Hazard rate path for firm i defines probability of default by time t, $Q_i(t)$
- Conditional on the sampled path the Q’s are independent and can be combined to determine the distribution of number of defaults (losses) at each time
Reduced Form Models - Comments

- To match Index on 14-Dec-06 requires:
  - Average hazard rate about 0.54%
- To match the 10-15% tranche on 14-Dec-06:
  - Only possible with common jumps
Structural Models

- A version of Merton’s model is implemented for each name in the portfolio
- Joint process for all asset values chosen
- Default barrier is chosen to match probability of default by time t for each name
  - Possibly a single t or for all t
  - Can match real (Moody’s) or implied risk-neutral default probabilities (Hull et al 2006))
- Monte Carlo simulation
Structural Models - Implementation

- **Strengths:**
  - Match individual firm characteristics
  - Processes as complex and realistic as you like
    - Stochastic volatility
    - Multi-factor
    - Jump-diffusion
    - Variable correlation models
  - Can value simple (CDO) or complex (CDO of CDO) products

- **Weaknesses:**
  - Slow
  - May be difficult to calibrate
If model is implemented so that asset values obey diffusion processes with a correlation structure determined by a single factor model:

\[
dV_i = \cdots dt + \sigma_i V_i \left[ a_i dZ_i + \sqrt{1 - a_i^2} dM \right]
\]

Results are almost the same as those from the Gaussian Copula, the market standard.
Factor Copulas – The Industry Standard

- Static ad hoc approach for correlating default probabilities
- Probability firm i defaults before t
  \[= \text{probability that } t_i < t = Q_i(t)\]
- Define credit quality variable for firm i, \(x_i\)
- Probability that credit quality \(x_i\) is less than \(x = F_i(x)\)
- Let \(Q_i(t) = F_i(x)\)
Credit Quality Model

- Credit quality determined by factor model

\[ x_i = \sqrt{\rho} M + \sqrt{1 - \rho} Z_i \]

\[ x_i < x | M \quad \text{if} \quad Z_i < \frac{x - \sqrt{\rho} M}{\sqrt{1 - \rho}} \]

\[
\text{Prob}(x_i < x | M) = H \left[ \frac{x - \sqrt{\rho} M}{\sqrt{1 - \rho}} \right]
\]

\[
\text{Prob}(t_i < t | M) = H \left[ \frac{F^{-1}\left(Q_i(t)\right) - \sqrt{\rho} M}{\sqrt{1 - \rho}} \right]
\]
Copula Implementation

- Conditional on M defaults are independent
- Distribution of number of defaults (losses) determined from the independent default probabilities
  - conditional value for pay and receive leg of tranche
- Integrate conditional values over distribution of M to get unconditional values
Copula Comments

- Usually use normal distributions for F and H
- Advantage:
  - Very fast and simple
  - matches each firm’s probability of default at each time
- Single unobserved variable, $\rho$, implied from tranche spreads
- Disadvantage:
  - implied correlation different for each tranche
Implied Copula Models

In copula model

\[ \text{Prob}(t_i < t \mid M) = H \left( \frac{F^{-1}(Q_i(t))x - \sqrt{\rho}M}{\sqrt{1-\rho}} \right) \]

 Defines a conditional default probability term structure for all \( t \) that occurs with probability \( G(M) \) \( dM \)

Implied copula model: choose set of conditional default probability term structures and choose probability

Can be fit to all tranches of a CDO simultaneously
Implied Copula Example

- Define default probability term structures by hazard rate
- Probability of default by $t$, $Q(t) = 1 - \exp(-\lambda t)$
- Hazard rate drawn from log-t distribution: mean $\mu$, standard deviation $\sigma$, $\nu$ degrees of freedom
- Three parameter model: choose $\mu$ to fit unconditional default probability and the other two parameters to fit the tranche spreads
Model Fit

Degrees of Freedom

Sigma

RMSE
Summary

- Except for the top down models all can be fitted to the characteristics of each individual firm.
- With enough degrees of freedom all can be simultaneously fitted to spreads for all tranches:
  - Except the market standard Gaussian copula.
- More economically intuitive (reduced form and structural models) tend to be more difficult to use.
- Gaussian copula more of a tool for conveying information:
  - Provides little intuition for results.