Dynamic Models of Portfolio Credit Risk: A Simplified Approach

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Portfolio Credit Derivatives

- Key product is a CDO
- Protection seller agrees to insure all losses on the portfolio that are between $X\%$ and $Y\%$ of the portfolio principal for life of contract (e.g. 5 yrs)
- Initial tranche principal is $(Y-X)\%$ of the portfolio principal
- Protection buyer pays a spread on the remaining tranche principal periodically (e.g. at the each quarter)
- Tranches of standard portfolios (iTraxx, CDX IG, etc) trade very actively
CDO models

- Standard market model is one-factor Gaussian copula model of time to default
- Alternatives that have been proposed: t-, double-t, Clayton, Archimedian, Marshall Olkin, implied copula
- All are static models. They provide a probability distribution for the loss over the life of the model, but do not describe how the loss evolves
Dynamic Models for Portfolio Losses: Prior Research

- **Structural**: Albanese et al; Baxter (2006); Hull et al (2005)
- **Reduced Form**: Duffie and Gârleanu (2001), Chapovsky et al (2006), Graziano and Rogers (2005), Hurd and Kuznetsov (2005), and Joshi and Stacey (2006)
Our Objective

- Build a simple dynamic model of the evolution of losses that is easy to implement and easy to calibrate to market data
- The model is developed as a reduced form model, but can also be presented as a top down model
CDO Valuation

- Key to valuing a CDO lies in the calculation of expected principal on payment dates
  - Expected payment on a payment date equals spread times expected principal on that date
  - Expected payoff between payment dates equals reduction in expected principal between the dates
  - Expected accrual payments can be calculated from expected payoffs
- Expected principal can be calculated from the cumulative default probabilities and recovery rates of companies in the portfolio
The Model (Homogeneous Case)

\[ dQ = \mu dt + dq \]

where \( Q \) is the an obligor’s cumulative default probability and \( dq \) represents a jump that has intensity \( \lambda \) and jump size \( h \)

\[ Q \quad \lambda \Delta t \quad 1-\lambda \Delta t \]

\[ Q + \mu \Delta t \quad Q + \mu \Delta t + h \]

\( \mu \) and \( \lambda \) are functions only of time and \( h \) is a function of the number of jumps so far. \( \mu > 0, \ h > 0, \) and \( Q \) is set equal to the minimum of 1 and the value given by the process
Implementation of Model

- Instruments such as CDOs, forward CDOs, and options on CDOs can be valued analytically.
- Model can be represented as a binomial tree to value other more complicated structures such as leveraged super seniors with loss triggers.
## Illustrative Data

Table 1  
**iTraxx CDO tranche quotes December 4, 2006.**

<table>
<thead>
<tr>
<th>$a_L$</th>
<th>$a_H$</th>
<th>3 yr</th>
<th>5 yr</th>
<th>7 yr</th>
<th>10 yr</th>
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<tr>
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<td>5.50</td>
<td>14.00</td>
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<td></td>
<td>12.69</td>
<td>24.75</td>
<td>32.83</td>
<td>43.59</td>
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</tbody>
</table>
Simplest Version of Model

- Jump size is constant and $\mu(t)$, is zero
- Jump intensity, $\lambda(t)$ is chosen to match the term structure of CDS spreads
- There is then a one-to-one correspondence between tranche quotes and jump size
- Implied jump sizes are similar to implied correlations
Comparison of Implied Jump Sizes with Implied Tranche Correlations

5-Year Quotes

7-Year Quotes

10-Year Quotes

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More Complex versions of the model. 

$\alpha(t) = \mu(t)/\mu_{\text{max}}(t)$. In all cases $\lambda(t)$ is chosen to fit CDS term structure.

- Constant $\alpha(t)$, constant jumps
- Constant $\alpha(t)$, size of $J$th jump, $h_J = h_0e^{\beta J}$. This provides a good fits to all tranches for a particular maturity.
- $\alpha(t)$ linear function of time, size of $J$th jump, $h_J = h_0e^{\beta J}$. This provides a good fit to all tranches for all maturities.
Variation of best fit $h_0$ and $\beta$ across time

Jump Parameters

$- h_0 \times 1,000$  $\beta$

4-Jul-06 3-Aug-06 2-Sep-06 2-Oct-06 1-Nov-06 1-Dec-06 31-Dec-06
Variation of best fit $\alpha(0)$ and $\alpha(10)$ across time
Evolution of Loss Distribution on Dec 4, 2006 for 4 parameter model.

Unconditional Loss Distribution at 4 Maturities

Probabilities for losses greater than 9% multiplied by 100.
Conclusions

- It is possible to develop a simple dynamic model for losses on a portfolio by modeling the cumulative default probability for a representative company.
- The only way of fitting the market appears to be by assuming that jumps in the cumulative default probability get progressively bigger.