Lévy Processes and Option Pricing

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A Lévy process is a continuous-time process that generates stationary, independent increments ... 

Think of return innovations ($\varepsilon$) in discrete time: 

$$R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}.$$ 

- Normal return innovation — diffusion
- Non-normal return innovation — jumps

Traditional Lévy specifications:
- either a Brownian motion (Black-Scholes)
- or a compound Poisson process with normal jump size (Merton).

⇒ The return innovation distribution is either normal or mixture of normals.
Lévy processes and return innovations

- Lévy processes greatly expand our continuous-time choices of iid return innovation distributions via the Lévy triplet \((\mu, \sigma, \pi(x))\). \((\pi(x)\)–Lévy density\).

- The Lévy-Khintchine Theorem:

\[
\phi_{X_t}(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)}, \\
\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}_0} \left(1 - e^{iu} + iux1_{|x|<1}\right) \pi(x)dx,
\]

Innovation distribution

\(\leftrightarrow\) characteristic exponent \(\psi(u)\)

\(\leftrightarrow\) Lévy triplet \((\mu, \sigma, \pi(x))\)

- Constraint: \(\int_0^1 x^2\pi(x)dx < \infty\) (finite quadratic variation).
Tractable examples

- Brownian motion \((\mu t + \sigma W_t)\): normal shocks.
- Merton’s compound Poisson jumps: Large but rare events.

\[
\pi(x) = \lambda \frac{1}{\sqrt{2\pi v_J}} \exp \left(-\frac{(x - \mu_J)^2}{2v_J}\right).
\]

- Dampened power law (DPL):

\[
\pi(x) = \begin{cases} 
\lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
\lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0,
\end{cases} \quad \lambda, \beta_\pm > 0, \quad \alpha \in [-1, 2)
\]

- Finite activity when \(\alpha < 0\): \(\int_{\mathbb{R}^0} \pi(x) dx < \infty\). Large and rare events.
- Infinite activity when \(\alpha \geq 0\): Both small and large jumps. Jump frequency increase with declining jump size, and approaches infinity as \(x \to 0\).
- Infinite variation when \(\alpha \geq 1\): many small jumps.
Analytical characteristic exponents

- Diffusion: $\psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2$.

- Merton’s compound Poisson jumps:
  \[ \psi(u) = \lambda \left( 1 - e^{iu\mu - \frac{1}{2}u^2v} \right). \]

- Dampened power law: ( for $\alpha \neq 0, 1$)
  \[ \psi(u) = -\lambda \Gamma(-\alpha) \left[ (\beta_+ - iu)^\alpha - \beta_+^\alpha + (\beta_- + iu)^\alpha - \beta_-^\alpha \right] \]
  - When $\alpha \to 2$, smooth transition to diffusion (quadratic function of $u$).
  - When $\alpha = 0$ (Variance-gamma by Madan et al):
    \[ \psi(u) = \lambda \ln \left( 1 - iu/\beta_+ \right) \left( 1 + iu/\beta_- \right). \]

- When $\alpha = 1$, exponentially dampened Cauchy, Wu (06):
  \[ \psi(u) = -\lambda \left( (\beta_+ - iu) \ln (\beta_+ - iu) / \beta_+ + \lambda (\beta_- + iu) \ln (\beta_- + iu) / \beta_- \right). \]
Other Lévy examples

- The normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1998)
- The generalized hyperbolic process (Eberlein, Keller, Prause (1998))
- The Meixner process (Schoutens (2003))
- All tractable in terms of the characteristic exponents $\psi(u)$.
- We can use FFT to generate the density function of the innovation (for model estimation).
- We can also use FFT to compute option values.
Run Brownian motions on a business clock

- Clark (1973): If one runs a Brownian motion on a business clock, the resulting process matches financial time series better.

- The possibility that business clock may not move while calendar time marches forward is important ...
  - A standard Poisson process $\Rightarrow$ the resulting process is a compound Poisson process with normal jump sizes.
  - A compound Poisson process with exponentially distributed jump size $\Rightarrow$ double-exponential compound Poisson process. (DPL with $\alpha = -1$)
  - A gamma process $\Rightarrow$ variance gamma (DPL with $\alpha = 0$).
  - A continuous clock $\Rightarrow$ a continuous process.
General evidence on Lévy return innovations

▶ Credit risk: \textbf{(compound) Poisson process}
  - The whole intensity-based credit modeling literature...

▶ Market risk: \textbf{Infinite-activity jumps}
  - Evidence from stock returns (CGMY (2002)): The $\alpha$ estimates for DPL on most stock return series are greater than zero.
  - Evidence from options: Models with infinite-activity return innovations price equity index options better (Carr and Wu (2003), Huang and Wu (2004))
Implied volatility smiles & skews on a stock

Maturities: 32, 95, 186, 368, 732

Moneyness = \ln(K/F) / \sigma \sqrt{\tau}

Short-term smile

Long-term skew
Implied volatility skews on SPX

More skews than smiles

Maturities: 32, 60, 151, 242, 333, 704
Average implied volatility smiles on currencies

Maturities: 1m (solid), 3m (dashed), 1y (dash-dotted)
Implied volatility smiles at short maturities

- Implied volatility smiles/skews ↔ non-normality/asymmetry for the underlying asset return risk-neutral distribution.

- Both jumps and stochastic volatility can generate return normalities, through different mechanisms.
  \[ R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1} \]
  - Jumps generate non-normality through the innovation distribution \((\varepsilon)\).
  - Stochastic volatility generates non-normality through mixing over multiple periods.

- Over short maturities (1 period), only jumps contribute to return non-normalities.
Time decay of short-term OTM options

- As option maturity ↓ zero, OTM option value ↓ zero.
- The speed of decay is exponential $O(e^{-c/T})$ under pure diffusion, but linear $O(T)$ in the presence of jumps.
- Term decay plot (Carr & Wu, 2003):
  \[ \ln(T) \sim \ln\left(\frac{OTM}{T}\right) \]
Central Limit Theorem (CLT) at long horizons

- CLT: As option maturity increases, the smile should flatten.
- Evidence: The skew does not flatten, but steepens!
- FMLS: Maximum negatively skewed $\alpha$-stable Lévy process.
  - Return variance is infinite. Hence, CLT does not apply.
  - All price moments are finite. Option has finite value.
- But CLT seems to hold fine statistically:

![Skewness on S&P 500 Index Return](image1)

![Kurtosis on S&P 500 Index Return](image2)
Reconcile $\mathbb{P}$ with $\mathbb{Q}$ via DPL

- Model return innovations under $\mathbb{P}$ by DPL:

  $$\pi(x) = \begin{cases} 
  \lambda \exp(-\beta_+ x) x^{-\alpha-1}, & x > 0, \\
  \lambda \exp(-\beta_- |x|) |x|^{-\alpha-1}, & x < 0.
  \end{cases}$$

  All return moments are finite with $\beta_{\pm} > 0$. *CLT applies.*

- Apply different market prices for up and down jumps:

  $$\frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_t = \exp(-\gamma^+ J^+ - \gamma^- J^- + \text{convexity adjustment})$$

- The return innovation process remains DPL under $\mathbb{Q}$:

  $$\pi(x) = \begin{cases} 
  \lambda \exp(-(\beta_+ + \gamma^+) x) x^{-\alpha-1}, & x > 0, \\
  \lambda \exp(-(\beta_- - \gamma^- |x|) |x|^{-\alpha-1}, & x < 0.
  \end{cases}$$

- To break CLT under $\mathbb{Q}$, set $\gamma^- = \beta_-$ so that $\beta^Q_- = 0$.

- Reconciling $\mathbb{P}$ with $\mathbb{Q}$: *Investors charge maximum allowed market price on down jumps.*
Default risk & long-term implied volatility skews

- When a company defaults, its stock value jumps to zero.
- It generates a steep skew in long-term stock options.
  - Default is really a first-moment effect: The pre-default risk-neutral drift is $r - q + \lambda_t$. CLT does not apply.
  - Using the second moment (implied vol) to capture the first-moment effect will generate large skews.
Capture Implied volatility smiles & skews with three (jump) components

I. **Market risk** (FMLS under $\mathcal{Q}$, DPL under $\mathbb{P}$)
   II. **Idiosyncratic risk** (DPL under both $\mathbb{P}$ and $\mathcal{Q}$)
   III. **Default risk** (Poisson arrival, jumps to zero).

▶ **Remarks:**
   ▶ Long-term implied volatilities are more correlated cross-sectionally than stock returns are.
   ▶ Market risk (I) is important. Identify (I) from SPX or QQQQ options.
   ▶ Default risk (III) is important for companies with low credit ratings (GM).
   ▶ Identify the credit risk component from the CDS market.

▶ **Currency:** The difference of two market risks.
Beyond Lévy processes

- Lévy processes can be used to generate different iid return innovation distributions.
- Yet, return distribution is iid, but varies stochastically over time.
- We need to go beyond Lévy processes to capture the stochastic nature of the return distribution.
Stochastic volatility on stock indexes

At-the-money implied volatilities at fixed time-to-maturities from 1 month to 5 years.
Stochastic volatility on currencies

Three-month delta-neutral straddle implied volatility.
Stochastic skewness on stock indexes

Implied volatility spread between 80% and 120% strikes at fixed time-to-maturities from 1 month to 5 years.
Three-month 10-delta risk reversal (blue lines) and butterfly spread (red lines).
Stochastically time-changed Lévy processes

- Discrete-time analog again: $R_{t+1} = \mu_t + \sigma_t \varepsilon_{t+1}$
  - $\varepsilon_{t+1}$ is an iid return innovation $\leftrightarrow$ Lévy process.
  - $(\mu_t, \sigma_t)$ can be time-varying, stochastic...

- If we start with a Lévy process, $(\mu, \sigma, \lambda \nu(x))$,
  $$
  \phi(u) \equiv \mathbb{E}[e^{iuX_t}] = e^{-t\psi(u)},
  \psi(u) = -iu\mu + \frac{1}{2}u^2\sigma^2 + \lambda \int_{\mathbb{R}_0} \left(1 - e^{iux} + iux1_{|x|<1}\right)\nu(x)dx,
  $$

- The drift $\mu$, the diffusion variance $\sigma^2$, and the arrival rate $\lambda$ are all proportional to time $t$.

- We can randomize the time $t \rightarrow T_t$ instead of randomizing $(\mu, \sigma^2, \lambda)$, for the same result.

- We define $T_t \equiv \int_0^t \nu_s^- ds$ as the (stochastic) time change, with $\nu_t$ being the instantaneous activity rate.
Model financial security returns for option pricing

- Start with the risk-neutral (\( \mathbb{Q} \)) process — That’s where tractability is needed the most dearly.
  - Identify the economic risk sources, model innovation on each source with a Lévy process (\( X^k_t \) for \( k = 1, \cdots, K \))
  - Apply separate time changes: \( X^k_t \rightarrow X^k_{T_t} \) to capture stochastic responses of financial security returns to economic shocks.

\[
\ln \frac{S_t}{S_0} = (r - q)t + \sum_{k=1}^{K} \left( b^k X^k_{T_t} - \varphi_x^k(b^k) I^k_{T_t} \right),
\]

- The framework makes model design more intuitive, parsimonious, and economically sensible.
  - Each Lévy component captures shocks from one economic source.
  - Time change captures the time-varying intensity of its impact.
Economic implications of using jumps

- **Black-Scholes (one-factor diffusion):**
  - The market is complete with a bond and a stock.
  - If you can estimate the statistical dynamics of the stock, you can price options on that stock.
  - Utility-free option pricing. Option prices are redundant. Options market reveals no extra information.

- **Heston (two-factor diffusion):** We can still complete the market with one extra option.

- **In the presence of jumps of random sizes,**
  - The market is inherently incomplete (with stocks alone).
  - Need all options (+ model) to complete the market.
  - Options market is informative/useful:
    - Cross-sectional behavior of options \((K, T) \leftrightarrow Q\) dynamics.
    - Time-series behavior of stocks/options \((t) \leftrightarrow P\) dynamics.
    - The difference \(Q/P \leftrightarrow\) market prices of economic risks.
Different types of jumps affect option pricing at both short and long maturities.

- Implied volatility smiles at very short maturities can only be accommodated by a jump component.
- Implied volatility skews at very long maturities ask for a jump process that generates infinite variance.
- Credit risk exposure may also help explain the long-term skew on single name stock options.

The choice of jump types depends on the modeled events:

- Infinite-activity jumps ⇔ frequent market order arrival.
- Finite-activity Poisson jumps ⇔ rare events (credit).

Applying stochastic time changes to the Lévy processes generates stochastic responses to each economic shock.

- generates stochastic volatility, skewness, ...

The presence of jumps of random sizes have important and practical applications for hedging...