Natural Selection in Financial Markets: Does It Work?

Hongjun Yan*

Yale School of Management

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Abstract

Can investors with incorrect beliefs survive in financial markets and have a significant impact on asset prices? My paper addresses this issue by analyzing a dynamic general equilibrium asset pricing model with some investors having rational expectations while others having incorrect beliefs concerning the mean growth rate of the economy. In contrast to the existing literature such as Sandroni (2000) and Blume and Easley (2004), which implicitly focus on economies without growth, this paper finds the elasticity of intertemporal substitution plays an important role for survival in a growth economy. The analysis also suggests that natural selection alone is not a satisfactory justification for rational expectations because i) it may take a long time to effectively eliminate the impact of irrational investors; ii) even small differences in preferences can make an investor dominate the market in the long run, even if his belief substantially and persistently deviates from the truth. The robustness of these results is discussed in various extensions of the baseline model.

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1 Introduction

The notion of rational expectations has been central to modern economics and finance ever since the pioneering work by Muth (1961), Radner (1972), Lucas (1978), and Cox, Ingersoll and Ross (1985). To justify the assumption that investors know the true distribution of the variables in the economy, economists have invoked the idea of natural selection (Alchian (1950), Enke (1951), and Friedman (1953)). The natural selection argument, also called market selection hypothesis, states that investors with incorrect beliefs will eventually be driven out of markets by those with rational expectations. Financial markets can therefore be understood, to a large extent, by models with only investors who have rational expectations.

In order to evaluate whether this natural selection mechanism can eliminate the impact of investors with incorrect beliefs, this paper analyzes a dynamic general equilibrium model of an exchange economy. The model has three main features. First, the previous literature has identified some conditions under which the rational or irrational investors can survive, but is silent on whether those conditions are plausible. To address this issue, I adopt a parametric specification of irrationality that allows me to calibrate the model to make assessments. Second, in the case where the investors with incorrect beliefs cannot survive in the limit, my calibrations reveal the time-span of the selection process, which is crucial in evaluating the effectiveness of the natural selection mechanism, and has not been studied by the previous literature. Third, while previous studies such as Sandroni (2000) and Blume and Easley (2004) implicitly focus on economies without growth, I consider an economy with and without growth, and show that the growth assumption reveals an important new insight about survival.

My analysis leads to the conclusion that the natural selection mechanism alone is unlikely to eliminate the impact of investors with incorrect beliefs or to provide a satisfactory foundation for the rational expectations assumption. There are two reasons for this. First, the selection process is excessively slow: in my benchmark calibrations, it takes, on average, hundreds of years for an investor to lose half of his wealth share, even if his beliefs persistently and significantly differ from the truth. Thus, investors with incorrect beliefs have a significant and long-lasting impact on asset prices, and models that ignore their
existence may fail to capture important dynamics of asset prices. Second, the natural selection process is very sensitive to small perturbations of preferences. A larger elasticity of intertemporal substitution (EIS hereafter) or a smaller time discount rate induces a higher saving motive, and so a better chance for survival. In fact, calibrations show that a slightly larger EIS or a slightly smaller time discount rate is sufficient for an investor to dominate the market, i.e. his wealth share approaches one almost surely when time goes to infinity, even if his beliefs persistently and substantially deviate from the truth.

This analysis also has an important implication for the role of irrationality in asset pricing, the main distinction between the behavioral finance and the traditional rational asset pricing paradigm. The recently developed behavioral finance has so far been based upon the notion of limits of arbitrage, which mainly relies on frictions for arbitrage such as agency issues and financial constraints (see Barberis and Thaler (2003) for a recent review). It is worth pointing out, however, that there is no arbitrage opportunity in my model. The advantage of having the correct belief is only gradually realized over time and the impact of irrationality on asset prices can be significant and long-lasting even in a financial market without any friction. These results further strengthen the argument that incorporating irrationality might help to improve our understanding of the behavior of asset prices.

To be more specific, I adopt a Lucas (1978)-type exchange economy with the aggregate dividend following a geometric Brownian motion (an assumption subsequently relaxed). The economy is populated by two long-lived investors with constant relative risk aversion (CRRA) utility functions. Investor 1 has rational expectations, while investor 2 has wrong beliefs concerning the mean growth rate of the aggregate dividend process. I assume that investor 2’s mistake is persistent, that is, he does not update his beliefs when observing additional data. This assumption makes it more difficult for investor 2 to survive, and therefore gives the natural selection mechanism a better chance to work. Both investors can trade in frictionless stock and bond markets to maximize, according to their beliefs, the expected discounted utility from their lifetime consumption.

The model is solved for economies with and without growth. While the implications from the non-growth case are similar to those in previous studies such as Sandroni (2000)
and Blume and Easley (2004), the growth economy case yields the following novel insights. First, when both investors have the same time discount rate and utility function, investor 2 cannot survive in the limit, that is, his wealth share approaches 0 almost surely when time goes to infinity. The intuition is as follows. An investor maximizes his expected utility by allocating more wealth to states that he believes are more probable. When an investor has incorrect beliefs, he allocates more wealth to states that he believes are more probable but are in fact not. Thus, on average, the investor with incorrect beliefs accumulates wealth at a lower rate, and so cannot survive in the limit. This result is similar to those in Sandroni (2000) and Blume and Easley (2004), with one noteworthy difference. In these two previous studies, which implicitly focus on economies without growth, one only need to control for the time discount rate to rule out irrational investors’ survival. In the growth economy case considered in current paper, however, one also has to control for the utility function. This will prove crucial later on.

Second, the parametric specification of irrationality allows me to calibrate the model to evaluate the time-span of the selection process, on which the previous literature is silent. Essentially, the rational investor’s only advantage is that he knows the mean economy growth rate. Due to risk aversion, however, the uncertainty in the economy prevents the rational investor from taking infinite positions. Consequently, the advantage of having the correct belief can only be realized over time. My calibrations reveal that for reasonable parameter values, the time-span of the selection process is “too long.” Suppose, for instance, that both investors have a relative risk aversion coefficient of 3; each investor has half of the total wealth at time 0; and investor 2 persistently overestimates the Sharpe ratio by 50%. Even after 100 years of evolution, investor 2’s expected wealth share is 42.3% and the probability for investor 2’s wealth to fall lower than 20% of the total wealth is merely 1.3%. Indeed, it takes, on average, around 400 years for investor 2 to lose half of his wealth share. Naturally, since investor 2 controls a large fraction of the total wealth, he has a significant and long-lasting impact on asset prices. After 100 years of evolution, the difference between the stock price in the economy with both investors and that in the economy with only

\footnote{Blume and Easley (2004) have some discussions on this issue, but their formulation, while allowing for great generality, makes it difficult to evaluate the time-span explicitly. In a contemporaneous study, Dumas, Kurshev and Uppal (2004) also find it takes a long time for an overconfident investor to lose his wealth share.}

\footnote{Similar results are obtained when investor 2 underestimates the Sharpe ratio.}
investor 1 still exceeds 10% on average. Taken together, these results suggest that although investor 2 cannot survive in the limit for this case, the effectiveness of the natural selection mechanism is largely undermined by the long time-span.

Third, I show that the selection result is very sensitive to small perturbations of preference parameters. Indeed, an investor is more likely to survive if he has a larger EIS or a smaller time discount rate. More importantly, according to my calibrations, a slightly larger EIS or a slightly smaller time discount rate is sufficient for an investor dominate the market even if his beliefs persistently and significantly deviate from the truth. Even if investor 2 persistently over- or underestimates the Sharpe ratio by 50%, he can still dominate the market in the limit if, for instance, his EIS is 0.683 and investor 1’s is 0.5, or if his time discount rate is smaller than investor 1’s by 0.0096.

The result on EIS is in contrast with previous studies such as Sandroni (2000) and Blume and Easley (2004), who find that the utility function is irrelevant for survival. The reason is as follows. EIS measures the sensitivity of an investor’s consumption growth to the economy growth. In the growth economy case in my model, the investor with a higher EIS prefers a higher consumption growth, and so has a higher saving rate and a better chance to survive. Note, however, that the economies considered in Sandroni (2000) and Blume and Easley (2004) have no growth and therefore EIS has no impact on the consumption growth and plays no role for survival. Indeed, their findings are consistent with my analysis for the case without growth.

The impact of the time discount rate is also intuitive. When an investor has a lower time discount rate, he discounts utility from future consumption less. He therefore chooses to save more and is more likely to survive. This has been noted in the previous literature (e.g. Blume and Easley (1992)). The contribution of this paper is to use the parametric specification of irrationality to demonstrate that even a slightly larger time discount rate might be enough to counter balance the disadvantage from a substantially wrong belief.

I examine the robustness of these results by extending the model to incorporate situations such as multiple stocks, multiple investors, general dynamics for the aggregate dividend process, and time-varying irrationality. The main insights are shown to be robust

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4 A simple analogy may be helpful for understanding this intuition. Suppose \( y = \alpha x \). That is, \( \alpha \) is the sensitivity of \( y \) to \( x \). In the case of \( x > 0 \) (analogous to the growth economy case), a higher \( \alpha \) leads to a higher \( y \). In the case of \( x = 0 \) (analogous to the non-growth economy case), however, \( \alpha \) has no impact on \( y \).
to these generalizations. Therefore, this paper concludes that the natural selection mechanism alone is unlikely to effectively eliminate the impact of investors with incorrect beliefs, and that irrationality can have a significant and long-lasting impact on asset prices even in a frictionless financial market without resorting to limits of arbitrage.

1.1 Related literature

Although the idea of natural selection in economics and finance dates back to the early 1950s, rigorous analyses are only recent. De Long, Shleifer, Summers and Waldman (DSSW hereafter) cast doubts on the idea of natural selection. DSSW (1990) argue that optimistic investors may earn a higher expected return than rational investors and DSSW (1991) show that optimistic investors may dominate the market in the limit in a model with exogenous prices. Blume and Easley (1992) analyze a model with endogenous asset prices but exogenous savings decisions. They point out that an investor with a logarithmic utility function chooses a strategy that maximizes the long-run wealth accumulation (LWA strategy hereafter). A rational investor with other utility functions optimally chooses to deviate from the LWA strategy. Thus, irrationality may be beneficial for survival if it happens to bring his portfolio choices closer to the LWA strategy. Kogan, Ross, Wang and Westerfield (KRWW 2004) analyze an economy with investors making portfolio decisions only, and show that a moderately optimistic investor can drive a rational investor out of the market if the relative risk aversion coefficient is larger than one. In contrast to these studies, Sandroni (2000) and Blume and Easley (2004) show that when investors make optimal decisions on both savings and portfolio choices, the investors with incorrect beliefs are always driven out of the market if financial markets are complete and all investors have the same time discount rate.

\[^5\text{In an early study, Penrose (1952) forcefully argues that it might not be proper to use biological analogies to understand firm behavior.}\]

\[^6\text{This is consistent with the intuition that in the case of the relative risk aversion coefficient being larger than one, a rational investor prefers to hold less stock than the LWA strategy. Moderate optimism induces an investor to hold more stock and so brings his strategy closer to the LWA strategy, and therefore helps him drive the rational investor out of the market. Similarly, it is straightforward to apply the analysis in KRWW (2004) to show that when the relative risk aversion coefficient is smaller than unity, a moderately pessimistic investor can drive the rational investor out of the market.}\]

\[^7\text{The incomplete-market case is studied by Blume and Easley (2004), Sandroni (2004) and Coury and Sciubba (2005). Other related studies consider issues related to imperfect competition (e.g. Palomino (1996), Kyle and Wang (1997), Benos (1998), Biais and Shadur (2000), Hirshleifer and Luo (2001)), asymmetric information (e.g. Mailath and Sandroni (2003), Sciubba (2004)), firm behavior (e.g. Luo (1995), Blume and}\]
The current paper also analyzes a general equilibrium model with endogenous savings and portfolio decisions. But in contrast with Sandroni (2000) and Blume and Easley (2004), which implicitly focus on economies no growth, I demonstrate that EIS is an important determinant for survival only in a growth economy. Moreover, my calibrations also assess the plausibility of the survival conditions for investors with incorrect beliefs and the time-span of the natural selection process, both of which are crucial for evaluating the natural selection argument and have not been explicitly studied in the previous literature.

This paper shares some similarities with KRWW (2004) in modeling the irrational investor’s beliefs. The key difference is that the investors in KRWW (2004) optimize over only portfolio decisions, while the investors in my model optimize over both savings and portfolio decisions. It is worthwhile pointing out that what is important is not the intermediate consumption; it is whether investors optimize over savings decisions that matters. In models where investors optimize over portfolio choice but not savings, e.g. Blume and Easley (1992) and KRWW (2004), wrong beliefs might be beneficial for survival, while in models where investors maximize over both, e.g. Sandroni (2000), Blume and Easley (2004), and the current paper, an incorrect belief is always a disadvantage for survival.

The remainder of the paper is organized as follows. In Section 2, I outline the benchmark economy with irrational investors. Section 3 characterizes the equilibrium in the presence of irrational investors. Section 4 discusses the implications for investors’ survival. Generalizations of the baseline model and further discussions are reported in Section 5. Section 6 concludes and the Appendix provides all proofs and illustrations.

2 Benchmark Economy with Irrational Investors

In this section, I adopt a simple model to demonstrate the main insights, most of which are also valid under more general conditions as will be shown in Section 5.
2.1 Information structure and investors’ perceptions

I consider a continuous-time pure-exchange economy with an infinite horizon. The uncertainty is represented by a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})\) on which is defined a one-dimensional Brownian motion \(Z(t)\). Let \(\{\mathcal{F}_t\}\) denote the augmented filtration generated by \(Z(t)\).

There is a single consumption good that serves as the numeraire. The exogenous aggregate dividend, or consumption supply, process \(D(t) > 0\) follows

\[
d\ln D(t) = g_D \, dt + \sigma_D \, dZ(t),
\]

where the mean growth rate \(g_D\) and the volatility \(\sigma_D\) are constants. For the rest of this paper, I will focus on the following two cases. The first case is for the growth economy, that is, \(g_D > 0\), which implies that the aggregate dividend grows over time and

\[
\lim_{t \to \infty} D(t) = \infty \quad a.s.
\]

While the second case is for the economy without growth, that is \(g_D = 0\), which implies that \(D(t)\) does not converge when time \(t\) goes to infinity.\(^8\)

There are two (types of) investors, investors 1 and 2. Although they observe the same dividend process, they may have different beliefs about its underlying structure. In particular, I assume that both investors can deduce the volatility \(\sigma_D\) from quadratic variation but have different beliefs about \(g_D\). I use \(g_D^1\) and \(g_D^2\), both are constants, to denote investors 1 and 2’s mean growth rate, respectively. This parametrization of the difference in beliefs is motivated by the insight from Merton (1980) that while the estimator of the variance converges to the true value when the data frequency goes to infinity, the estimator of the expected return does not converge to the true value for a finite estimation period.

Effectively, investor \(i\) \((i = 1, 2)\) is endowed with the probability space \((\Omega, \mathcal{F}^i, \{\mathcal{F}^i_t\}, \mathcal{P}^i)\). From investor \(i\)’s point of view, the dividend follows

\[
d\ln D(t) = g_D^i \, dt + \sigma_D \, dZ^i(t), \quad i = 1, 2,
\]

\(^8\)The \(g_D < 0\) case implies that \(\lim_{t \to \infty} D(t) = 0\) \(a.s.\). That is, the aggregate dividend decreases over time and converges to 0 in the limit. The analysis for such a contracting economy is similar to that for the growing economy studied in this paper. The implications for the contracting-economy case, omitted for brevity, are typically the mirror image of those for the growing economy.
where

\[ dZ^i(t) = dZ(t) - \frac{g^i_D - g_D}{\sigma_D} dt. \]  

(2)

By Girsanov’s theorem, \( Z^i(t) \) is a Brownian motion in the space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})\). The definition of \( dZ^i(t) \) in equation (2) implies

\[ dZ^2(t) = dZ^1(t) - \delta dt, \]  

where \( \delta \equiv \frac{g^2_D - g^1_D}{\sigma_D} \).

The parameter \( \delta \) measures the difference in beliefs between investors 1 and 2 regarding the mean dividend growth rate, normalized by its risk. For the rest of this paper, I will assume investor 1 has rational expectations, that is, \( g^1_D = g_D \), while investor 2 may have wrong beliefs, \( g^2_D \) may be different from \( g_D \). Thus, \( \delta \) can be interpreted as investor 2’s degree of irrationality. \( \delta \) is positive when investor 2 is more optimistic than investor 1, and negative when investor 2 is more pessimistic. This parametric specification for the degree of irrationality is important for calibrating the model to make quantitative assessments.

Note that \( \delta \) is a constant in this case. In other words, investor 2 does not update his beliefs when observing additional data and therefore his mistake is persistent. This assumption makes it more difficult for investor 2 to survive and gives the natural selection mechanism a better chance to work. In one of the extensions considered in Section 5, I also consider the case where investor 2’s degree of irrationality decreases over time. This can be interpreted as investor 2 learning over time from the data and his beliefs converging to the rational beliefs. Moreover, this paper considers only a particular form of irrationality, and uses the terms “irrational investor” and “investor with incorrect beliefs” interchangeably for expressional convenience.

### 2.2 Investment opportunity set

Investors can trade continuously a riskless bond and a stock. The bond is in zero net supply and the stock is a claim to the aggregate dividend process and has a net supply of one share. The bond price \( B(t) \), which is normalized so that \( B(0) = 1 \), and stock price \( S(t) \) have the following dynamics

\[
\begin{align*}
    dB(t) &= B(t) r(t) dt, \\
    dS(t) + D(t) dt &= S(t) [\mu(t) dt + \sigma(t) dZ(t)] \\
    &= S(t) [\mu^i(t) dt + \sigma(t) dZ^i(t)], \quad \text{for } i = 1, 2,
\end{align*}
\]

(3)

\[ 5 \]
where (3) represents the risky security dynamics as perceived by investor \( i \). All price coefficients are to be determined endogenously in equilibrium. Although investors have different beliefs about expected stock return, they agree on the stock price path, that is,

\[ \mu^1(t) dt + \sigma(t) dZ^1(t) = \mu^2(t) dt + \sigma(t) dZ^2(t), \]

which, together with (2), implies the following “consistency” relationship

\[ \mu^2(t) - \mu^1(t) = \sigma(t) \delta. \]  

(4)

The posited dynamic completeness of financial markets under the perceived price processes implies the existence of a unique state price density process for each investor \( \xi^i(t) \), consistent with no-arbitrage, given by

\[ d\xi^i(t) = -\xi^i(t) [r(t) dt + \kappa^i(t) dZ^i(t)], \]  

(5)

where \( \kappa^i(t) = \sigma(t)^{-1}(\mu^i(t) - r(t)) \) is the Sharpe ratio, or the market price of risk, process perceived by investor \( i \). The quantity \( \xi^i(t, \omega) \) is interpreted as the Arrow-Debreu price per unit probability \( P_i \) of a unit of consumption in state \( \omega \in \Omega \) at time \( t \) (with \( \xi^i(0) = 1 \)). The consistency condition (4) leads to

\[ \delta = \kappa^2(t) - \kappa^1(t). \]  

(6)

The above expression links investor 2’s degree of irrationality \( \delta \) to observable variables. Since investor 1 has rational expectations, that is, \( \kappa^1(t) = \kappa(t) \), investor 2’s degree of irrationality \( \delta \) is then the difference between his perception of the Sharpe ratio and the true one.

Finally, investors’ trading strategies are assumed to satisfy the following standard technical condition

\[ \int_0^T (\theta_i(t) S(t) \sigma(t))^2 dt < \infty, \]

for any \( T < \infty \), where \( \theta_i(t) \) is the number of shares of the stock held by investor \( i \) at time \( t \). This technical condition ensures that the stochastic integrals are well defined (Karatzas and Shreve (1988)).
2.3 Investors’ endowments and preferences

At time 0, investor \( i \) is endowed with \( \beta_i \) shares of the stock, with \( 0 \leq \beta_i \leq 1 \) and \( \beta_1 + \beta_2 = 1 \). He chooses a nonnegative consumption process \( c_i(t) \), and portfolio processes \( \theta_i(t) \), and so his financial wealth processes \( W_i(t) \) is

\[
dW_i(t) = W_i(t)r(t)dt - c_i(t)dt + \theta_i(t)S(t)[\mu_i(t) - r(t)]dt + \theta_i(t)S(t)\sigma(t)dz_i(t). \quad (7)
\]

Investor \( i \)'s utility function \( u_i(\cdot) \) is

\[
u_i(c(t)) = \frac{c(t)^{1-\gamma_i}}{1-\gamma_i} \quad \text{for } \gamma_i > 0, \quad (8)
\]

where \( \gamma_i \) is the relative risk aversion coefficient. The case of \( \gamma_i = 1 \) corresponds to the logarithmic utility function \( u_i(c(t)) = \log c(t) \). Note that for this constant relative risk aversion preference, the investor’s EIS is \( 1/\gamma_i \).

Investor \( i \)'s dynamic optimization problem is

\[
\max_{c_i} E^i_0 \left[ \int_0^{\infty} e^{-\rho_i t} u_i(c_i(t))dt \right], \quad (9)
\]

subject to the dynamic budget constraint (7), where \( \rho_i \) is the time discount rate and \( E^i_0 [\cdot] \) is the conditional expectation under the probability measure \( P^i \) conditional on the information set \( F_0^i \). In the rest of this paper, I confine my attention to cases where the expected utility is finite so that maximization problem (9) is well defined.

3 Equilibrium in the Presence of Irrational Investors

In this section, I solve for the equilibrium in the economy described above. The standard definition of competitive equilibrium in the presence of irrational investors is as follows.

**Definition 1** A competitive equilibrium is a price system \((r(t), S(t))\) and consumption-portfolio processes \((c_i(t), \theta_i(t))\) such that: (i) investors choose their optimal consumption-portfolio strategies given their perceived price processes; (ii) perceived security price processes are consistent across investors, i.e.,

\[
\mu^2(t) - \mu^1(t) = \sigma(t)\delta;
\]
and (iii) good and security markets clear, i.e.,

\[ c_1(t) + c_2(t) = D(t), \]
\[ \theta_1(t) + \theta_2(t) = 1, \]
\[ W_1(t) + W_2(t) = S(t). \]

To characterize the equilibrium, I adopt the technique developed in Basak (2000), who demonstrates that, in economies with heterogeneous beliefs, the equilibrium can be attained conveniently by constructing a representative investor with a stochastic weighting process, where the weighting process captures the difference of investors’ beliefs (see Basak (2004) for a recent review). Specifically, I define a representative investor with utility function

\[
U(c(t); \lambda(t)) \equiv \max_{c_1(t)+c_2(t)=c(t)} e^{-\rho_1 t} u_1(c_1(t)) + \lambda(t) e^{-\rho_2 t} u_2(c_2(t)),
\]

(10)

where \( \lambda(t) > 0 \) may be stochastic and captures the heterogeneity in beliefs.

One key observation from the formulation is that the weighting process is determined by the ratio of investors’ marginal utility:

\[
\lambda(t) = \frac{e^{-\rho_1 t} u_1'(c_1(t))}{e^{-\rho_2 t} u_2'(c_2(t))}.
\]

(11)

Thus, \( \lambda(t) \) is related to the ratio of investors’ discounted marginal utility at time \( t \) and so the asymptotic behavior of \( \lambda(t) \) reflects the investors’ survival in the limit, which will be formally defined in the next section. Moreover, from the martingale technique (Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989)), the optimality condition of the investor’s optimization problem implies that an investor’s marginal utility is proportional to his state price density. Thus equation (11) implies that \( \lambda(t) \) is proportional to \( \xi_1^1(t)/\xi_2^2(t) \). In the case of homogeneous investors, \( \xi_1^1(t) = \xi_2^2(t) \), and so \( \lambda(t) \) is a constant. In the case of heterogeneous beliefs, however, investors have different state price density processes and so \( \lambda(t) \) is stochastic and captures the difference in beliefs. Proposition 1 characterizes the equilibrium in closed form.

**Proposition 1** If equilibrium exists, the equilibrium interest rate is given by

\[
r(t) = -\frac{DU''(D(t); \lambda(t))}{U''(D(t); \lambda(t))},
\]

(12)
the stock price is given by
\[
S(t) = E_t \left[ \int_t^\infty \frac{U'(D(s); \lambda(s))}{U'(D(t); \lambda(t))} D(s)ds \right],
\] (13)
and the equilibrium consumption allocation is given by
\[
c_1(t) = I_1 \left( e^{\rho_1 t} U'(D(t); \lambda(t)) \right), \quad c_2(t) = I_2 \left( \frac{e^{\rho_2 t} U'(D(t); \lambda(t))}{\lambda(t)} \right),
\] (14)
where \( DU'(D(t); \lambda(t)) \) denotes the drift of the process \( U'(D(t); \lambda(t)) \), \( \lambda(t) \) satisfies
\[
d\lambda(t) = \lambda(t) \delta dZ(t),
\] (15)
\( \lambda(0) \) solves
\[
E \left[ \int_0^\infty U'(D(t); \lambda(t)) I_1 \left( e^{\rho_1 t} U'(D(t); \lambda(t)) \right) dt \right] = \beta_1 E \left[ \int_0^\infty U'(D(t); \lambda(t)) D(t) dt \right],
\] (16)
and \( I_i(\cdot) \) is the inverse of \( u_i'(\cdot) \) for \( i = 1, 2 \).

With the notion of the representative investor, the equilibrium interest rate and stock price can be characterized by the familiar valuation formula (equations (12) and (13)). Investors' optimal consumption, equation (14), can be obtained from their individual optimization problems in (9). Moreover, the dynamics of the weighting process \( \lambda(t) \) is characterized by exogenous variables in equation (15), with its initial value determined by equation (16). It is important to note that Proposition 1 holds also for any utility functions that are three times continuously differentiable, strictly increasing, strictly concave, and satisfy
\[
\lim_{c \to 0} u_i'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u_i'(c) = 0.
\]

Finally, note that the equilibrium allocation here is Pareto optimal. This is in contrast to the results for markets with frictions (Cuoco and He (1994), Basak and Cuoco (1998), Detemple and Serrat (2003)), which all construct equilibria by formulating a representative investor with stochastic weights. Appendix B.1 articulates why the stochastic weights are needed even though the markets are dynamically complete and the equilibrium allocation is Pareto efficient.

4 Implications on Survival

The main objective of this paper is to evaluate whether natural selection can effectively eliminate the impact of investors with incorrect beliefs on asset prices and justify the rational
expectations assumption. So, this section addresses the following two questions: First, which investor will survive in the long run? Second, if the investor with incorrect beliefs cannot survive, what is the time-span of the selection process? Denote investors’ consumption share as \( \omega_i(t) \), i.e. \( \omega_i(t) = c_i(t)/(c_1(t) + c_2(t)) \). The standard definition of survival and dominance are as follows.

**Definition 2** Investor \( i \) is said to become *extinct* if

\[
\lim_{t \to \infty} \omega_i(t) = 0, \quad a.s.;
\]

to *survive* if extinction does not occur; and to dominate the market if

\[
\lim_{t \to \infty} \omega_i(t) = 1, \quad a.s.
\]

The extinction, survival, and dominance of investor 2 is defined symmetrically.

An investor’s extinction is defined as his consumption share going to 0 in the limit. It is worthwhile pointing out that according to this definition, the consumption of the investor, who suffers from extinction in the limit, does not necessarily converge to 0. Indeed, an investor may not survive in the limit even if his consumption grows to infinity as long as the other investor’s consumption grows to infinity at an even higher rate.

**Proposition 2** In the economy described in Section 2, investors’ survival is given by

\[
\lim_{t \to \infty} \omega_2(t) = \begin{cases} 
0 & a.s. \text{ if } \frac{1}{2} \delta^2 > (\rho_1 - \rho_2) + (\gamma_1 - \gamma_2) g_D; \\
1 & a.s. \text{ if } \frac{1}{2} \delta^2 < (\rho_1 - \rho_2) + (\gamma_1 - \gamma_2) g_D.
\end{cases}
\]

Proposition 2 summarizes the main results on survival. It demonstrates that the survival result is determined by the tradeoff between investor 2’s degree of irrationality \( \delta \) and the difference in their preferences \((\rho_1 - \rho_2) + (\gamma_1 - \gamma_2) g_D\). Hence, there are three determinants for survival: beliefs, time discount rate and utility function. First, investor 2’s incorrect belief is a disadvantage for his survival. Investor 2 cannot survive if his degree of irrationality \( \delta \) is large enough, otherwise he can even drive the rational investor 1 out of the market. Second, the time discount rate determines an investor’s savings decision and so his chance for survival. Equation (17) demonstrates that investor 2 can even drive investor 1 out of the market if the difference in time discount rates, \( \rho_1 - \rho_2 \), is large enough. Third, the
utility function also plays a role for survival. Interestingly, this mechanism disappears in an economy without growth and this is clearly captured by equation (17): in the case without growth i.e. $g_o = 0$, the role of utility function disappears. The rest of this section elaborates on these three determinants in the following Corollaries 1–3, respectively.

4.1 The role of beliefs

To isolate the impact of investors’ beliefs, this section considers the case where investors 1 and 2 are identical in all aspects except for their beliefs, that is, $\rho_1 = \rho_2 = \rho$, $\gamma_1 = \gamma_2 = \gamma$, and $\delta \neq 0$. Sections 4.1.A and 4.1.B demonstrate that investor 2 cannot survive and has no impact on asset prices in the limit. Section 4.1.C, however, reveals the long time-span of this selection process: even when investor 2’s beliefs persistently and significantly deviate from the truth, after 100 years of evolution, he still controls a substantial fraction of the total wealth in the economy and has a considerable impact on asset prices.

4.1.A Survival in the limit

**Corollary 1 (Belief)** If both investors have the same time discount rate and risk aversion, i.e. $\rho_1 = \rho_2 = \rho$, $\gamma_1 = \gamma_2 = \gamma$, and investor 2 has incorrect beliefs, i.e. $\delta \neq 0$, then investor 2 cannot survive in the limit.

This corollary shows that when investors differ only in their beliefs the rational investor, investor 1, dominates the market in the limit. The intuition is the following. In the economy considered here, financial markets are dynamically complete, and therefore an investor can maximize his expected utility by allocating more wealth to states that he believes are more probable. Due to his wrong beliefs, investor 2 allocates more wealth to states that he believes are more probable but are actually not. Hence, investor 2’s wealth accumulates at a lower rate on average and cannot survive in the limit. Interestingly, it can be shown that, from investor 2’s perspective, he believes that his investment decision is better than that of investor 1’s, and that he will eventually drive investor 1 out of the market.

These results are similar to those in previous papers such as Sandroni (2000) and Blume and Easley (2004). However, as previously mentioned, both these papers implicitly focus on economies without growth. This leads to their conclusion that, after controlling for the time
discount rate, investors with incorrect beliefs will be driven out of the market irrespective of their utility functions. If we allow for growth, however, one has to control for both the time discount rate and the utility function to rule out the irrational investors, because, as will be demonstrated later, the utility function plays an important role for survival only in a growth economy.

It is worth pointing out that, similar to the implications in Sandroni (2000), Blume and Easley (2004), an incorrect belief is always a disadvantage for survival in my model. However, incorrect beliefs might be beneficial for survival in Blume and Easley (1992) and KRWW (2004). Blume and Easley (2004) pointed out that the reason is that the equilibrium in Blume and Easley (1992) is not Pareto optimal, and that an incorrect belief is always a disadvantage in a Pareto optimal equilibrium. However, the equilibrium in KRWW (2004) is Pareto optimal, but wrong beliefs can still be beneficial for survival. It turns out, this seeming inconsistency comes from a subtle difference in the interpretations of the models in KRWW (2004) and in other studies. See Appendix B.2 for more details.

4.1.B Price impacts in the limit

We are now ready to evaluate investor 2's price impacts in the limit. Let \( r^*(t) \) and \( S^*(t) \) denote the interest rate and the stock price in an economy with only investor 1 but otherwise identical to the economy described above. The Appendix shows that if both investors have the same time discount rate and risk aversion, then investor 2's impact disappears in the limit in the sense that

\[
\lim_{t \to \infty} \frac{r(t)}{r^*(t)} = 1 \text{ a.s.,} \tag{18}
\]

\[
\lim_{t \to \infty} \frac{S(t)}{S^*(t)} = 1 \text{ a.s.} \tag{19}
\]

These results are not surprising since Corollary 1 has shown that investor 2 cannot survive in the limit. However, KRWW (2004) provide an interesting example where an investor can have a finite impact on asset prices even when his wealth approaches 0, which suggests that the logical step from extinction to no-price-impact might not be straightforward.

In the example in KRWW (2004), the irrational investor with a trivial amount of wealth has a large impact on asset prices because he bets heavily on states where consumption is extremely valuable, i.e., the rational investor’s marginal utility approaches infinity. This
situation does not happen in the economy considered here because investor 2 maximizes his expected utility according to his beliefs. Hence, he does not bet heavily on the states where the aggregate dividend is almost 0. More precisely, from the lognormal distribution assumption for aggregate dividends, investor 2 believes that the probability for the aggregate dividend to be less than $\epsilon$ approaches 0 when $\epsilon$ goes to 0. Thus, he allocates a trivial fraction of his wealth to these states no matter how pessimistic he is, and his impact on asset prices disappears when his wealth share goes to 0.

4.1.C Price impacts after 100 years

It is important to note that there is no arbitrage opportunity in the economy. Investor 1’s only advantage is that he knows the true mean growth rate. Due to risk aversion, however, the uncertainty in the economy prevents him from taking infinite positions and hence investor 1’s advantage can only be realized gradually over time. The effectiveness of the natural selection argument then critically depends on the time-span of the selection process. A necessary condition for the natural selection argument to justify the rational expectations assumption is that irrational investors lose money to rational investors fairly “quickly.” In this case, the asset price behavior may largely be captured by models with only investors who have rational expectations. If the time-span of the selection process is “too long,” however, models that ignore the existence of these irrational investors may fail to capture important dynamics of current asset prices. I now evaluate whether 100 years is long enough for the selection mechanism to erode investor 2’s wealth share to a trivial fraction and whether the implications on asset prices are close to those from the model with the rational investor 1 only.

In particular, I conduct the following experiment. Suppose that investors 1 and 2 have the same wealth share at time 0. I then simulate the economy for 100 years and evaluate the irrational investor’s wealth share and impact on asset prices. Directly evaluating the expressions in Proposition 1 numerically is quite challenging. Fortunately, the equilibrium can be characterized explicitly as reported in the Appendix A (equations (53)–(57)) for cases where the relative risk aversion coefficient is an integer.

The simulation parameters are chosen as follows. Note that $\delta$, investor 2's degree of irrationality, is the difference between his estimate of the Sharpe ratio and the true one (as
expressed in equation (6)). For example, $\delta = 0.25\kappa$ implies that investor 2 overestimates the Sharpe ratio by 25%. In the following simulations, I consider the cases of $\delta = \pm 0.25\kappa$, $\delta = \pm 0.5\kappa$, $\delta = \pm 0.75\kappa$, and $\delta = \pm \kappa$. That is, investor 2 over- or underestimates the Sharpe ratio by 25%, 50%, 75%, or 100%. The dividend mean growth rate $g_D$ is set to match the historical consumption data. Thus, based on the estimate from U.S. data by Campbell (2003), I set $g_D = 1.789\%$ and $\kappa = 0.277$.

The economy is simulated 100,000 times. For each simulation path, I compute investor 2’s consumption and wealth share at time $T = 100$ years. Figure 1 reports the simulated probability density functions of investor 2’s consumption and wealth shares. Because investor 2 has a wrong belief, he is more likely to be poorer than investor 1. However, investor 2 still holds a significant fraction of total wealth even after he has persistently made mistakes for 100 years. For instance, when both investors have a relative risk aversion of 3, the upper-right plot of Panel B shows that, on average, investor 2 holds 42.3% of the total wealth in the economy even though he persistently overestimates the Sharpe ratio by 50% for 100 years. The probability that investor 2’s wealth share falls lower than 20% of total wealth is merely 1.3%. Indeed, with a probability of 29%, investor 2 can even be richer than investor 1. Naturally, with a large fraction of the wealth, investor 2 also consumes a large fraction of the total dividend, as shown in the upper-left plot.

Given that investor 2 holds a large fraction of the wealth after 100 years, it is not surprising that his existence has a large impact on asset prices. Investor 2’s impact on the stock price is captured by $\frac{S(T)}{S^0(T)}$, the ratio of the stock price in the economy considered here to that in the economy with only investor 1. The simulated probability density functions of $\frac{S(T)}{S^0(T)}$, reported in the lower-left plots of each panel, demonstrate investor 2’s large impact on the stock price. In the lower-left plot of Panel B, for example, when both investors have a relative risk aversion of 3, the average of investor 2’s impact on the stock price is around 10% ($\frac{S(T)}{S^0(T)}$ has a mean of 0.89). The result of $\frac{S(T)}{S^0(T)}$ being less than 1 implies that the existence of the optimistic investor 2 decreases the stock price. The reason is as follows. When the expected economy growth rate increases, it has two impacts on the stock valuation. On the one hand, the higher expected economy growth increases the stock price since the stock is the claim to future dividends of the whole economy. On the other hand, higher expected economy growth implies that future consumption is less valuable (lower
marginal utility from future consumption). To smooth consumption, investors prefer to hold less stock and consume more today, and so this drop in demand decreases the stock price. When investors’ relative risk aversion is larger than 1, the second effect dominates and therefore the existence of the optimistic investor 2 decreases the stock price. This intuition has long been noticed theoretically and might also be consistent with empirical evidence as argued in Yan (2003). (See Appendix B.3 for more details on this issue.) Finally, as well known in the literature (e.g. Basak (2004)), in the special case where both investors have logarithmic utility function, the existence of investor 2 has no impact on the stock price.

The lower-right plot of Panel B shows that the existence of investor 2 increases the riskfree interest rate by around 13% \( r(T) \) has a mean of 12.8%). This is also intuitive. One insight from models with homogeneous investors is that the interest rate is positively related with the expected economy growth rate (e.g. Lucas (1978) and Mehra and Prescott (1985)). This is because higher expected economy growth reduces investors’ saving motive and so increases the interest rate. When investors 1 and 2 have different beliefs, the interest rate is the weighted average of the interest rates that would obtain if investor 1 or 2 held all the wealth (Detemple and Murthy (1994)). That is, the interest rate is related to the weighted average of investors’ expected economy growth. Thus, the existence of optimistic investor 2 increases the interest rate.

Relative to the results in Panel B where investor 2 overestimates the Sharpe ratio by 50%, investor 2’s impact on asset prices is smaller when he overestimates the Sharpe ratio by 25%, as shown in Panel A of Figure 1. Similarly, Panels C and D show that investor 2’s impact is magnified when he overestimates the Sharpe ratio by 75% and 100%, respectively.

The previous simulation results are meant to demonstrate that it takes “too long” for the evolution process to eliminate the impact of irrational investors. However, what constitutes “too long” is subjective and may vary for different issues. To better evaluate the effectiveness of the natural selection mechanism, Figure 2 plots investor 2’s expected wealth share against time.

More specifically, the economic environment for this simulation is identical to that for Figure 1. At time 0, the economy is populated by investors 1 and 2 and each of them has a half share of the stock. For each time \( t = 50, 100, 150, \ldots, 1,000 \) years, the economy is simulated 1,000 times to compute the mean and standard deviation of investor 2’s wealth.
share. Since all the standard deviations are smaller than 1%, Figure 2 plots only investor 2’s expected wealth share against time $t$. Panel B, for example, corresponds to the case where investor 2 overestimates the Sharpe ratio by 50%. It shows that the expected half-life of investor 2’s wealth share is almost 400 years, if both investors have a relative risk aversion of 3. That is, it takes on average 400 years for investor 2 to lose half of his wealth share. The expected half-life of investor 2’s wealth share increases with $\gamma$. This is because higher risk aversion makes investors less aggressive in their trading in the stock market and therefore the mistake in beliefs erodes investor 2’s wealth share more slowly. Moreover, the comparison among Panels A–D, where investor 2 overestimates the Sharpe ratio by 25%, 50%, 75% and 100% respectively, reveals an intuitive pattern that expected half-life of investor 2’s wealth share decreases with respect to his degree of irrationality. In the case of $\gamma = 3$, for example, the expected half-life of investor 2’s wealth share is around 1,000, 400, 200, and 100 years in Panels A–D, respectively. The half-life of 100 years in Panel D, for example, implies that even if investor 2’s estimate of the Sharpe ratio is twice that of the true value, it still takes around 100 years for him to lose half of his wealth share. Finally, similar insights are obtained from simulations for various values of $\beta_1, g_D, \sigma_D, \rho$, and for cases where investor 2 is pessimistic. Pertinent plots are omitted here for brevity.

Finally, it is very important to note that my model inherits all the limitations of the standard consumption-CAPM, such as the difficulties in matching the equity premium and the stock return volatility. One might conjecture that the long time-span of the selection process is driven by the fact that the expected stock return in my model is very close to the risk free interest rate. To address this concern, I consider the following simple analysis. Suppose the stock price follows a geometric Brownian motion and the interest rate is a constant. Set the expected stock return and volatility and the interest rate according to the empirical data. I then solve the consumption and portfolio choice problem for investors with rational and irrational beliefs. Comparing the wealth dynamics of these two investors, I find the half-life of the irrational investor’s wealth share is similar to that in the above calibrations. \(^9\)

\(^9\)The details, omitted here for brevity, are available upon request.
4.2 The role of time discount rate

Intuitively, the time discount rate determines how patient an investor is. A higher time discount rate implies that the investor discounts utility from future consumption more heavily, and so is less willing to save. As a result, the investor with a higher time discount rate is less likely to survive. To demonstrate this intuition, I consider the case where both investors have the same utility function but are allowed to have different time discount rates.

**Corollary 2 (Time Discount Rate)** If both investors have the same risk aversion, then investors’ survival is given by

\[
\lim_{t \to \infty} \omega_2(t) = \begin{cases} 
0 & \text{a.s. if } \frac{1}{2} \delta^2 > \rho_1 - \rho_2, \\
1 & \text{a.s. if } \frac{1}{2} \delta^2 < \rho_1 - \rho_2.
\end{cases}
\]

This corollary reveals that investors’ survival is determined by the tradeoff between investor 2’s disadvantage from his incorrect beliefs, captured by $\delta$, and his advantage from his lower time discount rate, captured by $\rho_1 - \rho_2$. Investor 2 cannot survive if the impact from his wrong beliefs dominates ($\frac{1}{2} \delta^2 > \rho_1 - \rho_2$). However, when the impact from time discount rate is large enough ($\frac{1}{2} \delta^2 < \rho_1 - \rho_2$), investor 2 dominates the market in the limit despite his wrong beliefs.

The parametric specification of the degree of irrationality allows me to calibrate the model to evaluate whether or not the condition for investor 2 to survive, $\frac{1}{2} \delta^2 < \rho_1 - \rho_2$, is plausible. Note that $\delta$ represents the difference between investor 2’s perception of the Sharpe ratio and the true one (as expressed in equation (6)). If we set the Sharpe ratio as 0.277 (consistent with the estimate in Campbell (2003)), Table 1 shows, for example, even if investor 2 over- or underestimates the Sharpe ratio by 50%, he can still dominate the markets if his time discount rate is smaller than investor 1’s by 0.0096. It is important to note that economists’ view on time discount rate is widely dispersed. For example, in order to match certain properties of historical asset prices, previous calibrations in the literature have set $\rho$ to a wide range of values, ranging from $-0.1$ (Brennan and Xia (2001)), 0.0069 (Abel (2001)), 0.0202 (Barberis, Huang and Santos (2001)), 0.0695 (Abel (1999)) to 0.1165 (Campbell and Cochrane (1999)).

\footnote{The time discount rates for studies with discrete-time settings have been adjusted to make them comparable.}

10 Hence, the natural selection mechanism is very
sensitive to the time discount rate. Who the market selects may not be those with correct beliefs, but instead may simply be those with slightly lower time discount rates.

Corollary 2 is comparable to the result in Sandroni (2000) (Proposition 3) and Blume and Easley (2004) (Theorem 8). The contribution of this paper is to use the parametric specification of irrationality to demonstrate that the rational investor’s advantage for survival can be wiped out even by a slightly larger time discount rate. Moreover, the utility function is irrelevant for survival in these previous two studies. In the economy considered here, however, the utility function plays an important role for survival, as will be demonstrated next.

4.3 The role of the utility function

This section considers the case where both investors have the same time discount rate to demonstrate the impact of the utility function on survival. Setting \( \rho_1 = \rho_2 \) in Proposition 2, we obtain the following corollary.

**Corollary 3 (Utility function)** If both investors have the same time discount rate, i.e. \( \rho_1 = \rho_2 \), then investors’ survival is given by

\[
\lim_{t \to \infty} \omega_2(t) = \begin{cases} 
0 & \text{a.s. if } \frac{1}{2} \delta^2 > (\gamma_1 - \gamma_2) g_D; \\
1 & \text{a.s. if } \frac{1}{2} \delta^2 < (\gamma_1 - \gamma_2) g_D.
\end{cases}
\] (21)

This corollary shows that the survival is determined by the trade-off between investor 2’s degree of irrationality \( \delta \) and the difference in the utility functions \( \gamma_1 - \gamma_2 \). Equation (21) immediately leads to the following two examples.

**Example 1** In an economy without growth, i.e. \( g_D = 0 \), the utility function is irrelevant for survival and the investor with incorrect belief cannot survive.

This example shows that the utility function does not play a role for survival in this non-growth economy. It is important to note that, the conclusion in Example 1 holds also for any utility functions that are three times continuously differentiable, strictly increasing, strictly concave, and satisfy \( \lim_{c \to 0} u'_i(c) = \infty \) and \( \lim_{c \to \infty} u'_i(c) = 0 \). Thus, the findings in Sandroni (2000) and Blume and Easley (2004) are consistent with the implications in the non-growth case in my model.
It is also interesting to note that the dividend process in this case is not bounded. This observation suggests that boundedness is not necessary to rule out the role of utility functions for survival. As will be demonstrated later, it is whether the economy has growth that matters. In other words, the reason that utility functions play no role for survival in Sandroni (2000) and Blume and Easley (2004) is because there is no growth in their economies, not necessarily because they assume bounded dividend processes.

**Example 2** In a growth economy, i.e. $g_D > 0$, the investor with higher EIS would dominate the market in the limit if both investors are rational, i.e. $\delta = 0$.

This example demonstrates the advantage of a higher EIS for survival. The investor with lower EIS will be driven out of the market even though he also has correct beliefs.\(^{11}\) Intuitively, an investor’s preference for intertemporal substitution affects his consumption and savings decisions and so may affect his chances for survival in the limit. In particular, as discussed in the Introduction, EIS measures an investor’s motivation to smooth his consumption intertemporally: the higher the EIS, the more sensitive his consumption growth is to the growth of the economy. In this growing economy case, an investor with a higher EIS prefers to have a higher consumption growth rate, and so needs to save more to achieve this. As a result, the investor with a higher EIS has an advantage to survive in the limit.

For the CRRA utility case, there is a one-to-one relation between the relative risk aversion and EIS. Although the above intuition is consistent with the EIS argument, it is not possible, in principle, to rule out the role of the risk aversion without resorting to more general utility specifications such as the recursive utility (Epstein and Zin (1989) or stochastic differential utility (Duffie and Epstein (1992)). Nevertheless, it is very important to point out that the result in Example 2 holds also under certainty. In an economy without risk, i.e. $\sigma_D = 0$, the investor with lower EIS cannot survive in the limit. This observation reinforces the view that it is the EIS parameter that determines survival, since risk aversion plays no role in a certainty economy.

More generally, Corollary 3 illustrates the tradeoff between investor 2’s disadvantage from his wrong beliefs and his advantage from his higher EIS. On the one hand, investor

\(^{11}\)Similar results can be found in Dumas (1989), which considers a production economy with two rational investors having different relative risk aversion coefficients, and Wang (1996), which studies the term structure of interest rates in an exchange economy.
2’s wrong beliefs make his wealth grow at a lower rate than investor 1’s. The higher the degree of irrationality $\delta$, the larger the disadvantage. On the other hand, investor 2’s higher EIS induces him to save more. The larger the difference $\gamma_1 - \gamma_2$, the larger the advantage. When this advantage is large enough ($\frac{1}{2} \delta^2 < (\gamma_1 - \gamma_2) g_D$), the impact from his higher EIS dominates the impact from his wrong beliefs and so investor 2 dominates the market in the limit despite his wrong beliefs.

It is important to point out that the condition for investor 2 to dominate the markets in the limit, $\frac{1}{2} \delta^2 < (\gamma_1 - \gamma_2) g_D$, is also quite plausible. Setting $g_D = 1.789\%$ according to the estimate in Campbell (2003), Table 2 shows that even small difference in the utility function is enough to make investor 2 dominate the market in the limit for a large range of wrong beliefs. As an example, even if investor 2 over- or underestimate the Sharpe ratio by 50%, he can still dominate the market if $\gamma_1 - \gamma_2 > 0.54$, or if, for instance, investors 1 and 2 have an EIS of 0.5 and 0.683, respectively. Note that the empirical estimates of risk aversion are very dispersed, ranging from 0 to higher than 10, based on different datasets (see Chetty (2002) for a summary). Moreover, the empirical estimates of EIS are also quite dispersed. For instance, early evidence in Hall (1988) suggests EIS is close to 0, while recent evidence (Vissing-Jørgensen (2002), Vissing-Jørgensen and Attanasio (2003)), which explicitly takes into account the limited market participation, suggests heterogeneity among investors and that the EIS may even be larger than 1 for some investors. This suggests that natural selection is very sensitive to perturbations of EIS. Again, who the market selects may not be those with correct beliefs, but instead may simply be those with slightly higher EISs.

Finally, these results are in contrast with Sandroni (2000) and Blume and Easley (2004), who show that the utility function is irrelevant for investors’ survival. The reason is the following. The aggregate dividend process in these two previous studies is assumed to be uniformly bounded away from 0 and infinity. This assumption rules out the economy growth in the limit. As discussed in the Introduction, in a growth economy, a higher EIS induces a higher consumption growth rate, and this EIS impact disappears in an economy without growth. Thus, the boundness assumption in Sandroni (2000) and Blume and Easley (2004) rules out the impact of EIS on survival and leads to their finding, consistent with the result in Example 1, that the utility function plays no role for survival.
5 Generalizations and Discussions

This section generalizes the baseline model to incorporate multiple stocks, more than two investors, general dividend processes, and time-varying irrationality. Some further issues are also discussed.

5.1 Multiple stocks

When there is more than one stock, the irrational investor may overestimate the mean growth rates of some stocks and underestimate those of others. Hence, the irrational investor’s beliefs can deviate from the truth markedly even though his estimate of the mean growth of the aggregate economy is correct. This situation clearly cannot be captured by the baseline model. However, the analysis below shows that the main insights in Section 4 are still valid in the setting with multiple stocks.

The extension turns out to be straightforward. The only modification is to redefine the relevant quantities in the baseline model as their multi-dimensional counterparts. More precisely, let \( Z(t) \) denote an \( N \)-dimensional Brownian motion. Stocks \( 1, 2, \ldots, N \), are claims to \( N \) exogenous positive dividend processes \( D(t) = [D_1(t), D_2(t), \ldots, D_N(t)]^\top \) that follow

\[
\ln D(t) = g_D^i dt + \sigma_D^i dZ(i),
\]

where \( \ln D(t) = [\ln D_1(t), \ln D_2(t), \ldots, \ln D_N(t)]^\top \), \( g_D = [g_D(1), g_D(2), \ldots, g_D(N)]^\top \) is a \( N \times 1 \) constant vector, \( \sigma_D \) is a nonsingular \( N \times N \) constant matrix, and \( \gamma_D \geq 0 \), where \( \gamma_D = \max_{1 \leq k \leq N} g_D(k) \). From investor \( i \)'s point of view, the dividends follow

\[
\ln D(t) = g_D^i dt + \sigma_D^i dZ^i(t), \quad i = 1, 2,
\]

where

\[
dZ^2(t) = dZ^1(t) - \delta dt, \quad \delta \equiv \sigma_D^{-1} (g_D^2 - g_D^1).
\]

Investor 1 is assumed to have rational expectations. Thus, the \( N \)-dimensional vector \( \delta \) parameterizes investor 2’s degree of irrationality.
The price of a zero net supply bond, $B(t)$, which is normalized so that $B(0) = 1$, and stock prices $S(t) = [S_1(t), S_2(t), \ldots, S_N(t)]^\top$ have the following dynamics

$$dB(t) = B(t)r(t)dt,$$
$$dS(t) + D(t)dt = I_{S(t)}[\mu_i(t) dt + \sigma(t) dZ^i(t)], \text{ for } i = 1, 2,$$

where $I_{S(t)} = \text{diag}[S_1(t), S_2(t), \ldots, S_N(t)]$, the $N \times 1$ vector $\mu_i(t)$ are the expected returns perceived by investor $i$, and the volatility $\sigma(t)$ is a nonsingular $N \times N$ matrix. All the coefficients are to be determined in equilibrium.

By the same argument as in Section 2, that investors agree on the price path implies the following “consistency” relationship

$$\mu^2(t) - \mu^1(t) = \sigma(t)\delta. \tag{22}$$

The state price density process for each investor $i$, $\xi^i(t)$, is given by

$$d\xi^i(t) = -\xi^i(t)[r(t)dt + \kappa^i(t) dZ^i(t)],$$

where the $1 \times N$ vector $\kappa^i(t) = [\sigma(t)^{-1}(\mu^i(t) - r(t)1)^\top$, with $1$ denoting an $N \times 1$ vector of ones, is the Sharpe ratio perceived by investor $i$. Similar to the case with one stock, the consistency condition (22) implies

$$\kappa^2(t) - \kappa^1(t) = \delta,$$

where now $\delta$ is a vector.

The equilibrium is defined analogously as in Definition 1, and can be similarly characterized by constructing a representative investor as in equation (10). The implications for investors’ survival are summarized in the following proposition.

**Proposition 3** In the economy with $N$ stocks, investors’ survival is given by

$$\lim_{t \to \infty} \omega_2(t) = \begin{cases} 0 & \text{a.s. if } \frac{1}{2}\|\delta\|^2 > (\rho_1 - \rho_2) + (\gamma_1 - \gamma_2) g_D; \\ 1 & \text{a.s. if } \frac{1}{2}\|\delta\|^2 < (\rho_1 - \rho_2) + (\gamma_1 - \gamma_2) g_D. \end{cases} \tag{23}$$

The conditions for survival (equation (23)) are similar to those in the benchmark case (equation (17)) except that the degree of irrationality $\delta$ now becomes a vector. Therefore, the implications on the role of beliefs, the time discount rate and the utility function are similar to those in the benchmark case.
5.2 More than two investors

In this section, the only modification to the benchmark economy is to introduce $N > 2$ investors, who have CRRA utility functions

$$u_i(c) = \frac{c^{1-\gamma_i}}{1-\gamma_i},$$

for $i = 1, 2, \ldots, N$. There is still one stock which is a claim to the dividend process specified in equation (1). From investor $i$’s point of view, the dividend process follows

$$d\ln D(t) = g_D^i dt + \sigma_D dZ^i(t).$$

Thus, investor $i$ is effectively endowed with the probability space $(\Omega, \mathcal{F}^i, \{\mathcal{F}^i_t\}, \mathbb{P}^i)$ for $i = 1, \ldots, N$. To parameterize investor $i$’s degree of irrationality, define $\delta_i$ for $i = 1, \ldots, N$, as

$$\delta_i \equiv \frac{g_D^i - g_D}{\sigma_D}. \tag{24}$$

I assume that investor 1 has correct beliefs, that is $\delta_1 = 0$. For the same argument as in Section 2, investor $i$’s perceived Sharpe ratio satisfies

$$\delta_i = \kappa^i(t) - \kappa^1(t), \quad i = 1, \ldots, N.$$

To solve for the equilibrium, I define the representative investor’s utility as

$$U(c; \lambda) \equiv \max_{c_1, \ldots, c_N} \sum_{i=1}^N \lambda_i(t)e^{-\rho_i t}u_i(c_i) \tag{25}$$

subject to $c_1 + \ldots + c_N = c$, where $\lambda \equiv [\lambda_1, \ldots, \lambda_N]$. Similar to the baseline model, the weighting processes can be identified by investors’ marginal utilities

$$\lambda_i(t) = e^{(\rho_i - \rho_1)t} \frac{u_1'(c_1(t))}{u_i'(c_i(t))}, \quad i = 1, \ldots, N \tag{26}$$

and $\lambda_i(0)$ satisfies investor $i$’s binding budget constraint, and the weighting process of investor $i$ is given by

$$d\lambda_i(t) = \lambda_i(t)\delta_i dZ(t), \quad i = 1, \ldots, N \tag{27}$$

For $i = 1, 2, \ldots, N$, define investor $i$’s survival index, $I_i$, as

$$I_i = \rho_i + \gamma_i g_D + \frac{1}{2} \delta_i^2. \tag{28}$$

The following proposition reports the results on investors’ survival in the limit.
Proposition 4 In an economy with \( N \) investors, only the investor(s) with the lowest survival index can survive.

This proposition reveals that an investor cannot survive if there exists another investor with a lower survival index. If there are several investors having the lowest survival index, they all survive in the limit. Moreover, similar to the baseline model, Proposition 4 clearly reveals that a higher time discount rate, a lower EIS, and a higher degree of irrationality are disadvantages for survival.

5.3 General dividend process

The analysis so far assumes the dividend process as a geometric Brownian motion. The following analysis, however, shows that the main insights from the baseline model are also valid for general dividend processes. More specifically, the only modification to the baseline model here is to replace the dividend process in equation (1) by a general process

\[
d\ln D(t) = g_D(t)\, dt + \sigma_D(t)\, dZ(t),
\]

where the volatility processes is assumed to be \( \{F^D_t\} \)-adapted, and \( \{F^D_t\} \) denotes the filtration generated by \( D(t) \). That is, both investors 1 and 2 know the volatility process \( \sigma_D(t) \). Moreover, investor 1 can also observe the mean growth \( g_D(t) \) but investor 2 mistakenly believes the mean growth rate to be \( g_2^D(t) \). The degree of irrationality \( \delta \) is

\[
\delta = \frac{g_2^D(t) - g_D(t)}{\sigma_D(t)},
\]

which is assumed to be a constant. The financial markets are as described in Section 2 and the representative investor can be constructed as in equation (10). The results on investors’ survival are reported in the following proposition.

Proposition 5 In the economy with a general dividend process given by equation (29), investors’ survival is given by

\[
\lim_{t \to \infty} \omega_2(t) = \begin{cases} 0 & \text{a.s. if } \lim_{t \to \infty} e^{(\rho_1 - \rho_2 - \frac{1}{2} \delta^2)t} D(t) \gamma_1 - \gamma_2 = 0 \text{ a.s.;} \\ 1 & \text{a.s. if } \lim_{t \to \infty} e^{(\rho_1 - \rho_2 - \frac{1}{2} \delta^2)t} D(t) \gamma_1 - \gamma_2 = \infty \text{ a.s.} \end{cases}
\]
Equation (30) reveals that the tradeoff between the incorrect belief and the difference in preferences determines who will survive, in a way similar to that in Proposition 2. In fact it is easy to verify that equation (30) collapses to equation (17) when the dividend process \( D(t) \) is a geometric Brownian motion.

5.4 Time-varying degree of irrationality

The analysis so far takes the irrational investors' degree of irrationality as a constant. However, one can imagine that an investor’s degree of irrationality may change over time for various reasons, such as learning. This section demonstrates some generalizations to incorporate the time variation in the degree of irrationality. In particular, the aggregate dividend process and the financial markets are the same as in the baseline model. I focus on the case where both investors have the same time discount rate and utility function. Investor 1 has rational expectations while investor 2’s beliefs of the mean dividend growth \( g_{2D}(t) \) is assumed to be time-varying. Similarly, I define investor 2’s degree of irrationality

\[
\delta(t) \equiv \frac{g_{2D}(t) - g_D}{\sigma_D}.
\]  

(31)

I also assume that \( \delta(t) \) is uniformly bounded

\[ |\delta(t)| < K \text{ for any } t, \]  

(32)

where \( K \) is a constant. Note that \( K \) can be an arbitrarily large constant. That is, the deviation of investor 2’s beliefs from the truth has an arbitrarily large upper bound. Hence, the assumption (32) does not lose much economic generality.

Instead of deriving investor 2’s beliefs from certain learning rules, I assume an exogenous process \( \delta(t) \) and suppose that

\[
\lim_{t \to \infty} \delta(t) = \epsilon.
\]  

(33)

That is, investor 2’s degree of irrationality changes over time and converges to \( \epsilon \) in the limit. Thus, \( \epsilon = 0 \) corresponds to the case where investor 2’s belief converges to the rational belief eventually, and \( \epsilon \neq 0 \) implies that investor 2 is never able to learn the truth. The following proposition characterizes investors’ survival.
Proposition 6 Suppose that investors 1 and 2 have the same time discount rate and utility function, and that investor 1 has rational expectations and investor 2 has a time-varying degree of irrationality given by equations (31)–(33).

- In the case of $\epsilon \neq 0$, investor 2 will be driven out of the market in the limit

$$\lim_{t \to \infty} \omega_2(t) = 0 \text{ a.s.}; \quad (34)$$

- In the case of $\epsilon = 0$, if there exists a constant $\alpha > 1/2$ and $t_0$ such that

$$|\delta(t)| < \frac{1}{t^\alpha} \text{ for all } t > t_0 \text{ a.s.}, \quad (35)$$

then both investors survive.

Investor 2 will be driven out of the market if his beliefs do not converge to the truth (equation (34)). The intuition for this is similar to that in Proposition 2: investor 2’s wealth grows at a lower rate since he consistently makes mistakes on the mean dividend growth rate. More interestingly, investor 2 may become extinct in the limit even if his beliefs eventually converge to the truth. This is because, as long as $\delta(t) \neq 0$, investor 2 is making inferior savings and portfolio decisions and so loses wealth share on average. If he learns very slowly, that is, $\delta(t)$ converges to 0 very slowly, his wealth share will eventually go to 0. One sufficient condition for investor 2 to survive is given by equation (35), that is, if the degree of irrationality converges to 0 faster than $\frac{1}{t^\alpha}$ does where $\alpha > \frac{1}{2}$. To see an example where investor 2 cannot survive even though his beliefs converge to the rational one, set $\delta(t) = \frac{1}{t^{1/3}}$. In this case, $\lim_{t \to \infty} \delta(t) = 0$, that is, investor 2’s beliefs converge to the truth as $t$ goes to infinity. Yet, Appendix B.4 shows that investor 2 becomes extinct in the limit since he learns too slowly.

5.5 The interaction between rationality and preferences

Previous analysis treats preferences as an exogenous factors and so rules out the interaction between rationality and preferences. While the detailed analysis on this issue is beyond the scope of this paper, the following conjecture suggests this might be an interesting area for future research.
Suppose the economy is populated by investors with different degrees of irrationality. The more sophisticated investors, i.e., those whose beliefs are closer to the truth, may eventually become fund managers. As Shleifer and Vishny (1997) point out, institutional investors may have incentives to care more about the short-term performance than normal individual investors. This can be interpreted as that the time discount rates of those sophisticated investors increase due to the agency problems. This increase in time discount rates counteracts the advantages that those sophisticated investors originally had for survival. In other words, it becomes more difficult for the economy to evolve toward rationality. Further exploration of the implication from this interaction might be fruitful.

5.6 One implication for questionnaire investigations

One version of the market selection hypothesis is that investors who behave irrationally will eventually be driven out of the market by those who behave as if they are rational. That is, the investors selected by the market are those whose behavior happens to be consistent with rationality, and who are not necessarily rational per se. As summarized in Blume and Easley (1992), “without assuming much about the behavior of individual economic actor, evolutionary forces drive the economy towards a state which can be described by the conventional neoclassical models with optimizing actors.” That is, the natural selection mechanism implies that models with assumptions of maximization and equilibrium can make useful predictions although the agents in the models do not literally solve those complex maximization problems as assumed.

One important implication from this argument is that the effectiveness of questionnaire investigations is doubtful since investors do not really solve the maximization problems. For instance, Friedman (1953, p. 31) comments that testing economic theories by “answers given by businessmen to questions about the factors affecting their decisions . . . is about on a par with testing theories of longevity by asking octogenarians how they account for their long life.” The analysis in this paper, however, demonstrates that the natural selection forces might not be effective in driving the economy towards the state which can be described by models with only rational investors. Therefore, my analysis undermines the validity of the argument against the questionnaire investigations based on natural selection.
5.7 The recursive utility case: an open issue

In settings with exogenous savings decision, e.g. Blume and Easley (1992), wrong beliefs can be beneficial for survival if it happens to bring an investor’s portfolio strategy closer to the log investor’s strategy. In Sandroni (2000), Blume and Easley (2004) and the current paper, this becomes impossible because the savings decision is endogenous. What will happen in the setting where investors optimally choose to have a constant consumption wealth ratio? More precisely, consider a setting with two investors having the same Epstein-Zin recursive utilities with an EIS of 1. As Giovannini and Weil (1989) have shown, these investors optimally choose to consume a constant proportion of their wealth. Who would survive in this case?

One natural conjecture from the insight in Blume and Easley (1992) is that the modest optimistic (pessimistic) investor would dominate if their relative risk aversion coefficient is larger (smaller) than 1. If this is the case, it would suggest that the conclusion from Blume and Easley (2004), that wrong beliefs cannot be beneficial for survival in a Pareto optimal equilibrium, cannot be extended to the case of general preferences. Note that the analysis in Blume and Easley (2004) is carried out with time separable preferences. So, the open questions are: Does the insight in Blume and Easley (2004) hold for time non-separable preferences? Who will survive when investors have recursive preferences? I leave these open issues to future research.

6 Conclusion

Charles Darwin’s evolutionary theory is one of the most influential theories in biology, and economists have long been tempted by its intuitive argument. This paper nevertheless demonstrates some limitations of this idea in a competitive financial market setting.

First, the long time-span of the selection process might be less of an issue for biologists. Many biological phenomena can be attributed to thousands of years of evolution. As Paul Krugman (1996) comments: “evolutionary theorists normally take the shortcut of assuming that the process gets you to the maximum, and pay surprisingly little attention to the dynamics along the way.” For financial economists, however, the long time-span of the evolution process is more problematic. As this paper has demonstrated, it takes hundreds
of years for an investor to lose half of his wealth share, even if his beliefs persistently and significantly differ from the truth. The existence of irrational investors, then, has a significant and long-lasting impact on asset prices. Thus, models that ignore these irrational investors may fail to capture important dynamics of asset prices.

Second, the outcome of the natural selection process is sensitive to small perturbations of preferences: The higher saving motive induced by a slightly larger elasticity of intertemporal substitution or a slightly smaller time discount rate can make an investor dominate the market in the limit even if his beliefs persistently and substantially deviate from the truth. In other words, the survivors of the natural selection process may not be the investors with correct beliefs but simply those with higher saving motives.

Based on these results from the baseline model, and similar results from its generalizations that incorporate multiple stocks, multiple investors, general dividend processes, and time-varying degree of irrationality, this paper concludes that the natural selection mechanism alone is unlikely to effectively eliminate the impact of investors with incorrect beliefs or to provide a satisfactory foundation for the assumption of rational expectations.

It is also important to note that the recently developed behavioral finance literature has so far been based upon the notion of limits of arbitrage, which argues that the effectiveness of arbitrage activities may be limited due to agency issues or financial constraints. The results in this paper, however, suggest that irrationality might play an important role for asset pricing, both at finite horizons and asymptotically, even if the financial markets are frictionless and there is no limits of arbitrage. These results further suggest the importance of incorporating irrationality into asset pricing models.

It is also worth pointing out that the main point in this paper is that the natural selection is unlikely to be effective in ruling out the role of irrationality for asset pricing. The arguments in this paper are not against the idea of natural selection or evolution in finance and economics in general. On the contrary, showing that the role of irrationality may not be ruled out effectively, my analysis reinforces, rather than undermines, the importance of searching for alternatives for the rational expectations paradigm. See Lo (2004) for a recent proposal based on the idea of evolution.
Appendix A: Proofs

Proof of Proposition 1

With a representative investor constructed in (10), the state price density process is identified by the representative investor’s marginal utility $U'(D(t); \lambda(t))$. This leads to the interest rate and stock price expressions (12) and (13).

Using martingale techniques (Cox and Huang, 1989; Karatzas et al., 1987), each investor’s dynamic optimization problem can be rewritten as a static one:

$$\max_{c_i} E^i \left[ \int_0^\infty e^{-\rho t} u_i(c_i(t)) dt \right]$$

subject to

$$E^i \left[ \int_0^\infty \xi^i(t) c_i(t) dt \right] \leq \beta_i E^i \left[ \int_0^\infty \xi^i(t) D(t) dt \right].$$

The necessary and sufficient conditions for optimality of the consumption streams are

$$e^{-\rho t} u_i'(c_i(t)) = y_i \xi^i(t), \quad i = 1, 2,$$

where $y_i > 0$ is such that investor $i$’s static budget constraint holds with equality at the optimum, i.e. $y_i$ satisfies

$$E^i \left[ \int_0^\infty \xi^i(t) I_i(e^{\rho t} y_i \xi^i(t)) dt \right] = \beta_i E^i \left[ \int_0^\infty \xi^i(t) D(t) dt \right], \quad i = 1, 2.$$}

The optimal consumption (14) follows from equation (38). Applying Ito’s lemma to (11) with (38) substituted in, we obtain (15). $\lambda(0)$ can be pinned down by investor 1’s static budget constraint (16).

Proof of Proposition 2 and Corollaries 1–3

Substituting (8) into (11), we obtain

$$\frac{e^{-\rho_1 t} c_1(t)^{-\gamma_1}}{e^{-\rho_2 t} c_2(t)^{-\gamma_2}} = \lambda(t),$$

which implies

$$\frac{\omega_2(t)^{\gamma_2}}{\omega_1(t)^{\gamma_1}} = e^{(\rho_1 - \rho_2)^t} \lambda(t) D(t)^{\gamma_1 - \gamma_2},$$

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where \( \omega_1(t) = c_1(t)/D(t) \) and \( \omega_2(t) = c_2(t)/D(t) \). After some algebra, equation (41) leads to

\[
\frac{\omega_2(t)^{\gamma_2}}{\omega_1(t)^{\gamma_1}} = \lambda(0)D(0)^{\gamma_1-\gamma_2}e^{\left[(\rho_1-\rho_2)+\gamma_1-\gamma_2)g_D-\frac{1}{2}\sigma_D^2+[\rho+(\gamma_1-\gamma_2)\sigma_D]\frac{Z(t)}{t}\right]}.
\]

From the strong law of large numbers (see Karatzas and Shreve 1988 p. 104), we have

\[
\lim_{t \to \infty} \frac{Z(t)}{t} = 0 \quad a.s.
\]

In the case of \((\rho_1-\rho_2) + (\gamma_1-\gamma_2)g_D - \frac{1}{2}\sigma^2 > 0\), we obtain

\[
\lim_{t \to \infty} \frac{\omega_2(t)^{\gamma_2}}{\omega_1(t)^{\gamma_1}} = \infty, \quad a.s.,
\]

and therefore

\[
\lim_{t \to \infty} \omega_2(t) = 1, \quad a.s.
\]

Similarly, in the case of \((\rho_1-\rho_2) + (\gamma_1-\gamma_2)g_D - \frac{1}{2}\sigma^2 < 0\), we obtain

\[
\lim_{t \to \infty} \omega_2(t) = 0, \quad a.s.
\]

Substitution of pertinent parameters yields Corollaries 1–3.

**Proof of Proposition Equations (18) and (19)**

In the economy with only the rational investor, the interest rate and stock price are

\[
r^*(t) = \rho + \gamma \mu_D - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2,
\]

\[
S^*(t) = D(t)E_t \left[ \int_t^{\infty} e^{-\rho(s-t)} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma} ds \right].
\]

Under the condition \( \rho_1 = \rho_2 = \rho \), \( \gamma_1 = \gamma_2 = \gamma \), the construction of the representative investor in (10) leads to

\[
U'(D(t); \lambda(t)) = e^{-\rho t} u_1'(c_1(t)).
\]

Equation (40), together with the market clearing condition \( c_1(t) + c_2(t) = D(t) \), implies

\[
c_1(t) = \frac{D(t)}{1 + \lambda(t)^{1/\gamma}},
\]

Substituting (44) and (45) into (12) and (13), we obtain

\[
r(t) = \rho + \gamma \left( \frac{1}{1 + \lambda(t)^{1/\gamma}} g_D + \frac{\lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} g_D^2 \right) - \frac{1}{2} \gamma^2 \sigma_D^2 + \frac{\gamma - 1}{2\gamma} \frac{\lambda(t)^{1/\gamma}}{(1 + \lambda(t)^{1/\gamma})^2} \delta^2.
\]
and
\[
S(t) = \frac{D(t)^\gamma}{(1 + \lambda(t)^{1/\gamma})^\gamma} E_t \left[ \int_t^\infty e^{-\rho(s-t)}(1 + \lambda(s)^{1/\gamma})^\gamma D(s)^{1-\gamma} ds \right].
\] (47)

Hence, equations (42) and (46) lead to (18).

We can rewrite \((1 + \lambda(s)^{1/\gamma})^\gamma\) as its Taylor expansion with a Lagrange remainder:
\[
(1 + \lambda(s)^{1/\gamma})^\gamma = 1 + \gamma [\lambda^*(s)]^{1/\gamma},
\] (48)

where
\[
0 \leq \lambda^*(s) \leq \lambda(s).
\] (49)

Substituting (48) into (47), after some manipulation, we obtain
\[
\frac{S(t)}{D(t)} = \frac{1}{(1 + \lambda(t)^{1/\gamma})^\gamma} \left[ \frac{S^*(t)}{D(t)} + E_t \left[ \int_t^\infty e^{-\rho(s-t)} \gamma [\lambda^*(s)]^{1/\gamma} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma} ds \right] \right].
\] (50)

Hence
\[
\frac{S(t)}{D(t)} \geq \frac{1}{(1 + \lambda(t)^{1/\gamma})^\gamma} \frac{S^*(t)}{D(t)}.
\]

Multiplying \(\frac{D(t)}{S^*(t)}\) to both sides and letting \(t\) goes to infinity, we obtain
\[
\lim_{t \to \infty} \frac{S(t)}{S^*(t)} \geq 1.
\] (51)

Moreover, (49) and (50) imply
\[
\frac{S(t)}{D(t)} \leq \frac{1}{(1 + \lambda(t)^{1/\gamma})^\gamma} \left[ \frac{S^*(t)}{D(t)} + E_t \left[ \int_t^\infty \gamma e^{-\rho(s-t)} e^{-\frac{1}{2}\delta^2 t + \frac{\delta}{2} Z(t)} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma} ds \right] \right].
\]

After some algebra, we obtain
\[
\frac{S(t)}{S^*(t)} \leq \frac{1}{(1 + \lambda(t)^{1/\gamma})^\gamma} \left[ 1 + \lambda(t)^{1/\gamma} \frac{D(t)}{S^*(t)} E_t \left[ \int_t^\infty \gamma e^{-\rho(s-t)} e^{-\frac{1}{2}\delta^2 t + \frac{\delta}{2} Z(t)} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma} ds \right] \right].
\]

Letting \(t\) goes to infinity, we obtain
\[
\lim_{t \to \infty} \frac{S(t)}{S^*(t)} \leq 1.
\] (52)

Equations (51) and (52) lead to (19).
Corollary 4 If both investors have the same time discount rate and risk aversion, i.e. 
$\rho_1 = \rho_2 = \rho$, $\gamma_1 = \gamma_2 = \gamma$, and $\gamma$ is an integer, then the interest rate is given by
\[
r(t) = \rho + \gamma \left( \frac{1}{1 + \lambda(t)^{1/\gamma} g_D} + \frac{\lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma} g_D^2} \right) - \frac{1}{2} \gamma^2 \sigma_D^2
\]
\[+ \frac{\gamma - 1}{2 \gamma} \frac{\lambda(t)^{1/\gamma}}{(1 + \lambda(t)^{1/\gamma})^2} \delta^2,
\]
the stock price is given by
\[
S(t) = \sum_{k=0}^{\gamma} \binom{\gamma - 1}{k} a_k \lambda(t)^{k/\gamma} D(t),
\]
investor 1’s consumption share is given by
\[
c_1(t) = \frac{1}{D(t)} \frac{1}{1 + \lambda(t)^{1/\gamma}},
\]
and investor 1’s wealth share is given by
\[
W_1(t) = \frac{\sum_{k=0}^{\gamma-1} \binom{\gamma - 1}{k} a_k \lambda(t)^{k/\gamma}}{\sum_{k=0}^{\gamma} \binom{\gamma}{k} a_k \lambda(t)^{k/\gamma}},
\]
where
\[
d\lambda(t) = \lambda(t) \delta dZ,
\]
$\lambda(0)$ solves the polynomial equation
\[
\sum_{k=0}^{\gamma-1} \binom{\gamma - 1}{k} a_k \lambda(0)^{k/\gamma} = \beta_1 \sum_{k=0}^{\gamma} \binom{\gamma}{k} a_k \lambda(0)^{k/\gamma},
\]
and, for $k = 0, 1, 2, \ldots, \gamma$, $\binom{\gamma}{k}$ is a binomial constant and the constant $a_k$ is given by
\[
a_k = \left[ \rho + \frac{1}{2} k \left( 1 - \frac{k}{\gamma} \right) \delta^2 + (\gamma - 1) \left( g_D - \frac{1}{2} (\gamma - 1) \sigma_D^2 + \delta \frac{k}{\gamma} \sigma_D \right) \right]^{-1}.
\]

Comment: With the extra assumption of $\gamma$ being an integer, the equilibrium characterized in Proposition 1 is considerably simplified. In particular, the interest rate, stock price, and investor 1’s wealth share can be computed explicitly as functions of exogenous parameters (equations (53)–(56)) and the equation for $\lambda(0)$ is simplified to a polynomial equation (see equation (57)). These results greatly simplify the simulations in this paper.
Proof: The interest rate expression (53) has been obtained in (46). To prove (54), we note that, when $\gamma$ is an integer, $(1 + \lambda(s)^{1/\gamma})^\gamma$ can be expanded as

$$(1 + \lambda(s)^{1/\gamma})^\gamma = \sum_{k=0}^{\gamma} \binom{\gamma}{k} \lambda(s)^{k/\gamma}, \quad (58)$$

where, for $k = 0, 1, 2, \ldots, \gamma$, $\binom{\gamma}{k}$ is a binomial constant. Substituting the above expression into (47), we obtain

$$S(t) = \frac{D(t)^\gamma}{(1 + \lambda(t)^{1/\gamma})^\gamma} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \sum_{k=0}^{\gamma} \binom{\gamma}{k} \lambda(s)^{k/\gamma} D(s)^{1-\gamma} ds \right].$$

$$= \frac{D(t)}{(1 + \lambda(t)^{1/\gamma})^\gamma} \sum_{k=0}^{\gamma} \binom{\gamma}{k} \int_t^\infty e^{-\rho(s-t)} E_t \left[ \lambda(s)^{k/\gamma} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma} \right] ds. \quad (59)$$

Note that $\lambda(s)^{k/\gamma} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma}$ is lognormally distributed, then $E_t \left[ \lambda(s)^{k/\gamma} \left( \frac{D(s)}{D(t)} \right)^{1-\gamma} \right]$ can be evaluated explicitly. Substituting it into (59), after some algebra, we get (54).

Similarly, investor 1’s wealth is the present value of his future consumption flow $c_1(t)$:

$$W_1(t) = E \left[ \int_t^\infty e^{-\rho(s-t)} \frac{u_1'(c_1(s))}{u_1'(c_1(t))} c_1(s) ds \right].$$

Substituting (45) into the above expression we obtain

$$W_1(t) = \frac{D(t)^\gamma}{(1 + \lambda(t)^{1/\gamma})^\gamma} E \left[ \int_t^\infty e^{-\rho(s-t)} \left( 1 + \lambda(s)^{1/\gamma} \right)^{\gamma-1} D(s)^{1-\gamma} ds \right]. \quad (60)$$

Note that $(1 + \lambda(s)^{1/\gamma})^{\gamma-1}$ can be written as

$$\left( 1 + \lambda(s)^{1/\gamma} \right)^{\gamma-1} = \sum_{k=0}^{\gamma-1} \binom{\gamma-1}{k} \lambda(s)^{k/\gamma}. \quad (61)$$

Substituting the above expression into (60), after some similar computation, we obtain

$$W_1(t) = \frac{D(t)^\gamma}{(1 + \lambda(t)^{1/\gamma})^\gamma} \sum_{k=0}^{\gamma-1} \binom{\gamma-1}{k} a_k \lambda(t)^{k/\gamma},$$

and so get (56). Equation (55) immediately follows from (45).

Substituting (44) and (45) into (16), we obtain

$$E \left[ \int_0^\infty e^{-\rho(s-t)} \left( 1 + \lambda(t)^{1/\gamma} \right)^{\gamma-1} D(t)^{1-\gamma} dt \right] = \beta_1 E \left[ \int_0^\infty e^{-\rho(s-t)} \left( 1 + \lambda(t)^{1/\gamma} \right) D(t)^{1-\gamma} dt \right].$$

Substituting (58) and (61) into the above expression, after some similar computation, we obtain (57).
Proof of Proposition 3

Without loss of generality, assume $g_D(1) \geq g_D(2) \geq \ldots \geq g_D(N)$. That is, stock 1 has the highest growth rate: $g_D(1) = \overline{g}_D$. Equation (41) implies

$$\frac{\omega_2(t)^{\gamma_2}}{\omega_1(t)^{\gamma_1}} = e^{(\rho_1 - \rho_2)t} \lambda(t) \left( \sum_{i=1}^{N} D_i(t) \right)^{\gamma_1 - \gamma_2}$$

$$= e^{(\rho_1 - \rho_2)t} \lambda(t) D_1(t)^{\gamma_2 - \gamma_1} \left( \sum_{i=1}^{N} \frac{D_i(t)}{D_1(t)} \right)^{\gamma_1 - \gamma_2}, \quad (62)$$

In the case of $\frac{1}{2} \| \delta \|^2 > (\rho_1 - \rho_2) + (\gamma_1 - \gamma_2) \overline{g}_D$, applying Taylor expansion with a Lagrange remainder, we obtain

$$\frac{\omega_2(t)^{\gamma_2}}{\omega_1(t)^{\gamma_1}} = e^{(\rho_1 - \rho_2)t} \lambda(t) D_1(t)^{\gamma_2 - \gamma_1} \left( 1 + \sum_{i=2}^{N} \frac{D_i(t)}{D_1(t)} \right)^{\gamma_1 - \gamma_2}$$

$$= e^{(\rho_1 - \rho_2)t} \lambda(t) D_1(t)^{\gamma_2 - \gamma_1} \left( 1 + (\gamma_1 - \gamma_2) \sum_{i=2}^{N} \frac{D^*_i(t)}{D_1(t)} \right), \quad (63)$$

where $0 \leq D^*_i(t) \leq D_i(t)$, for $i = 1, 2, \ldots, N$. Note that

$$\lim_{t \to \infty} e^{(\rho_1 - \rho_2)t} \lambda(t) D_1(t)^{\gamma_2 - \gamma_1} = 0, \quad a.s. \quad (64)$$

$$\lim_{t \to \infty} e^{(\rho_1 - \rho_2)t} \lambda(t) D_1(t)^{\gamma_2 - \gamma_1} \frac{D^*_i(t)}{D_1(t)} = 0, \quad a.s. \quad \text{for } i = 1, 2, \ldots, N. \quad (65)$$

Equations (63)–(65) imply

$$\lim_{t \to \infty} \frac{\omega_2(t)^{\gamma_2}}{\omega_1(t)^{\gamma_1}} = 0, \quad a.s.$$

and therefore,

$$\lim_{t \to \infty} \omega_2(t) = 0, \quad a.s.$$

In the case of $\frac{1}{2} \| \delta \|^2 < (\rho_1 - \rho_2) + (\gamma_1 - \gamma_2) \overline{g}_D$, we obtain

$$\frac{\omega_1(t)^{\gamma_1}}{\omega_2(t)^{\gamma_2}} = e^{(\rho_1 - \rho_2)t} \lambda(t) D_1(t)^{\gamma_2 - \gamma_1} \left( 1 + \sum_{i=2}^{N} \frac{D_i(t)}{D_1(t)} \right)^{\gamma_2 - \gamma_1}.$$ 

Applying similar argument, we obtain

$$\lim_{t \to \infty} \omega_2(t) = 1, \quad a.s.$$

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Proof of Proposition 4

The optimality of investor $i$’s consumption implies

$$e^{-\rho_i t} u'_i(c_i(t)) = y_i \xi^i(t),$$

where $y_i > 0$ is a constant and $\xi^i(t)$ is investor $i$’s state price density process. Applying Ito’s lemma to (26), we obtain (27). Substituting (8) into (26), we obtain for any $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$,

$$
e^{-\rho_1 t} c_1(t)^{-\gamma_1} = \lambda_i(t), \quad e^{-\rho_i t} c_i(t)^{-\gamma_i} = \lambda_j(t).
$$

(66)

Suppose there is an investor $i$ having a survival index lower than investor $j$: $I_i < I_j$. By the same argument as in the proof of Proposition 2, we obtain

$$
\frac{\omega_j(t)^{\gamma_j}}{\omega_i(t)^{\gamma_i}} = e^{(\rho_i - \rho_j) t} \frac{\lambda_j(t)}{\lambda_i(t)} D(t)^{\gamma_i - \gamma_j}.
$$

Therefore, $\lim_{t \to \infty} e^{(\rho_i - \rho_j) t} \frac{\lambda(t)}{\lambda_i(t)} D(t)^{\gamma_i - \gamma_j} = 0$, a.s. implies $\lim_{t \to \infty} \omega_j(t) = 0$, a.s.

We can easily verify that if investors $i$ and $j$ have the same lowest survival index, neither $\omega_i(t)$ nor $\omega_j(t)$ converges to 0.

Proof of Proposition 5

Equation (41) immediately leads to the result in (30).

Proof of Proposition 6

The weighting process $\lambda(t)$ has a dynamic of

$$d\lambda(t) = \lambda(t) \delta(t) dZ(t),$$

and can be identified by their marginal utility

$$\lambda(t) = \frac{c_2(t)^{-\gamma}}{c_1(t)^{-\gamma}}.$$

We then obtain

$$\frac{c_2(t)}{c_1(t)} = \lambda(t)^{1/\gamma} = \lambda(t_0)^{1/\gamma} e^{-g(t)},
$$

(67)
where
\[
g(t) = \frac{1}{\gamma} \left[ \frac{1}{2} \int_{t_0}^{t} \delta(s)^2 ds - \int_{t_0}^{t} \delta(s) dZ(s) \right], \tag{68}
\]

We can rewrite \( g(t) \) as
\[
g(t) = \frac{1}{\gamma} f(t) (t - t_0),
\]
where
\[
f(t) = \frac{1}{2} \frac{1}{t - t_0} \int_{t_0}^{t} \delta(s)^2 ds - \frac{1}{t - t_0} \int_{t_0}^{t} \delta(s) dZ(s) \tag{69}
\]

Equations (32) and (33) lead to
\[
\lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \delta(s)^2 ds = \epsilon^2 \text{ a.s.} \tag{70}
\]

Moreover, \( \frac{1}{t - t_0} \int_{t_0}^{t} \delta(s) dZ(s) \) is normally distributed with mean 0 and variance
\[
V(t) = \frac{1}{(t - t_0)^2} E \int_{t_0}^{t} \delta^2(s) ds.
\]

Condition (32) implies
\[
V(t) \leq \frac{1}{(t - t_0)^2} \int_{t_0}^{t} K^2 ds = \frac{K^2}{t - t_0},
\]
and hence
\[
\lim_{t \to \infty} V(t) = 0.
\]

As a result,
\[
\lim_{t \to \infty} \frac{1}{t - t_0} \int_{t_0}^{t} \delta(s) dZ(s) = 0 \text{ a.s.} \tag{71}
\]

Equations (69)–(71) imply that
\[
\lim_{t \to \infty} f(t) = \frac{1}{2} \epsilon^2,
\]
and
\[
\lim_{t \to \infty} g(t) = \infty,
\]
which leads to (34).

The condition (35) implies that
\[
\int_{t_0}^{t} \delta(s)^2 ds < \int_{t_0}^{t} \frac{1}{t^{2a}} ds.
\]

Thus, if \( \alpha > 1/2 \), we have
\[
\lim_{t \to \infty} \int_{t_0}^{t} \delta(s)^2 ds < \infty \text{ a.s.}
\]
Moreover, $\int_0^t \delta(s) dZ(s)$ is a normal random variable with mean 0 and variance $E \int_0^t \delta(s)^2 ds$, which is finite. That is, when $t$ goes to infinity, $g(t)$ has finite mean and variance. Hence both investors survive in the limit.
Appendix B: Illustrations

B.1 Why is the stochastic weighting process needed?

The stochastic weights capture the difference in beliefs, and are *not* contradictory to the fact that a representative investor with constant weight can be constructed if the equilibrium allocation is Pareto optimal. The difference is that the constant weight is “outside” the expectation operator while the stochastic weight constructed in equation (10) is “inside.”

From the Second Welfare Theorem, the Pareto optimality implies that the equilibrium can also be generated by a representative investor with the following maximization problem with a *constant weight* \( \lambda^* \)

\[
\max_{c_1(t) + c_2(t) = c(t)} E^1 \left[ \int_0^{\infty} e^{-\rho_1 t} u_1(c_1(t); t) dt \right] + \lambda^* E^2 \left[ \int_0^{\infty} e^{-\rho_2 t} u_2(c_2(t); t) dt \right]. \tag{72}
\]

Note that the weight \( \lambda^* \) is outside of the expectation operator \( E^2 \). In fact, this formulation is essentially the same as the representative investor constructed in equation (10) since the above formulation, equation (72), can be written as

\[
\max_{c(t)} E^1 \left[ \int_0^{\infty} V(c(t); \lambda(t)) dt \right],
\]

where

\[
V(c(t); \lambda(t)) = \max_{c_1(t) + c_2(t) = c(t)} e^{-\rho_1 t} u_1(c_1(t); t) + \lambda^* \frac{dP^2}{dP^1}(t) e^{-\rho_2 t} u_2(c_2(t); t)
\]

and \( \frac{dP^2}{dP^1}(t) \) is the density process of the probability measure \( P^2 \) with respect to \( P^1 \). Comparing this with equation (10) we have

\[
\lambda(t) = \lambda^* \frac{dP^2}{dP^1}(t).
\]

Though essentially identical to the constant weight formulation, the formulation with the stochastic weight process \( \lambda(t) \) greatly simplifies the analysis here because it fully captures the difference in beliefs and, more importantly, can be pinned down by exogenous quantities.
B.2 Why can wrong beliefs be beneficial for survival in KRWW (2004), where the equilibrium is Pareto optimal?

Blume and Easley (2004) point out an important insight that “Pareto optimality is the key to understanding selection for or against traders with rational expectations.” Essentially, this is the reason why an incorrect belief is always a disadvantage for survival in Sandroni (2000), Blume and Easley (2004), and the current paper, in which the equilibria are Pareto optimal, and why an incorrect belief might be an advantage for survival in Blume and Easley (1992), in which the equilibrium is not Pareto optimal.

Interestingly, even though the equilibrium in KRWW (2004) is Pareto optimal, an incorrect belief can still be an advantage for survival. This seemingly contradictory result in fact comes from a subtle difference in their model. KRWW (2004) analyzes a finite horizon economy, and then take the horizon $T$ to infinity to study the implications on survival. Here is the subtle point. When we change the horizon from $T_1$ to $T_2$, we are actually dealing with two different economies. In other words, when $T$ changes, the economic setting itself also changes, and so investors’ portfolio strategy etc. changes. In contrast, in Sandroni (2000), Blume and Easley (2004), and the current paper, the dynamics of economic environment is specified first and will not change afterwards.

To be more specific, the Proposition 1 in KRWW (2004) shows (in their notations) that

$$b \xi_T = \left( \frac{c_{r,T}}{c_{n,T}} \right)^{-\gamma},$$

where $b$ is the “constant” weight of the central planner; $\xi_T$ is the Radon-Nikodym derivative; $c_{r,T}$ and $c_{n,T}$ are the consumption of the rational and irrational investors; and $\gamma$ is the relative risk aversion coefficient. Note that this weight $b$ depends on the horizon $T$ (see equation (10) in KRWW (2004)). When the economic setting changes (i.e., $T$ changes), so does the weight $b$, and in fact $b$ converges to $-\infty$, 1, or $\infty$ depends on the sign of $(\gamma - 1)\eta$. In Sandroni (2000), Blume and Easley (2004), and the current paper, however, the weight ($\lambda^*$ in equation (72)) is a constant and never changes since the economic setting does not change. It is the dependence of $b$ on $T$ in KRWW (2004) that induces the seeming contradiction. Note that in the special case of $\gamma = 1$, $b$ does not depend on $T$ (see equation (10) in KRWW (2004)). In this case, as expected, wrong beliefs are always disadvantages for survival.
B.3 Expected economy growth and stock valuation

It has long been recognized that the stock valuation (e.g. price dividend ratio) decreases when the expected economy growth rate increases in an exchange economy when the representative investor has a relative risk aversion higher than 1 (e.g. Cochrane (2001) p. 48). The intuition is as follows. When the expected economy growth increases, it has two impacts on the stock valuation. First, the higher expected economy growth increases the stock price since the stock is the claim to future dividends from the whole economy. Second, the higher expected economy growth implies lower marginal utility in the future. To smooth consumption, the investor prefers to consume more today and so hold less stock. This lower demand for stock decreases its price. When the representative investor’s relative risk aversion is larger than 1, the second effect dominates and the price dividend ratio decreases in the expected economy growth. In fact, this property can be found in many previous asset pricing models, see, for example, equation (6) in Mehra and Prescott (1985), equation (20) in Abel (1988), equations (7) through (12) in Cecchetti, Lam and Mark (1990), equations (8) through (13) in Cecchetti, Lam and Mark (1993), equation (7) in Veronesi (2000), equation (4) in Whitelaw (2000) and equation (21) in Yan (2001).

Yan (2003) points out that the above intuition relies on the assumption that the stock is the claim to the aggregate dividend of the economy. When this assumption is relaxed, the second effect is generally much weaker for individual stocks and the tradeoff between these two effects leads to a variety of new implications. For example, his model implies that, at the individual stock level, the correlation between stock returns and earnings surprises is generally positive. At the aggregate level, however, this correlation is smaller or even negative. This is consistent with the empirical evidence at both the individual stock level (Ball and Brown (1968)) and the aggregate level (Kothari, Lewellen and Warner (2004)). Another implication, which has also been confirmed by the empirical evidence documented in Yan (2003), is that a stock’s return is less sensitive to its earnings surprises if its earnings growth is more correlated with the aggregate consumption growth. Finally, when a stock accounts for a sufficiently large fraction of the economy, the decrease of its information quality decreases its risk premium as found in Veronesi (2000) with only one risky asset. However, when the stock is only a small fraction of the economy, as observed in reality, the decrease of its information quality increases its risk premium.
B.4 An example where an investor cannot survive even if his beliefs converge to the truth

To come up with such an example, one need to make $\delta(t)$ converge to 0 at a very low speed. One example is $\delta(t) = t^{-\alpha}$, with $\alpha < 1/2$. Clearly, $\lim_{t \to \infty} \delta(t) = 0$. The following, however, shows that investor 2 cannot survive in this case.

Substituting $\delta(t) = t^{-\alpha}$ into equation (67) in the proof of Proposition 13, we obtain

$$\frac{c_2(t)}{c_1(t)} = \lambda(t)^{1/\gamma} = \lambda(t_0)^{1/\gamma} e^{-h(t)t^{1-2\alpha}}, \quad (73)$$

where

$$h(t) = \frac{1}{t^{1-2\alpha}} \frac{1}{2} \frac{1}{1 - 2\alpha} \int_{t_0}^t s^{-2\alpha} ds - \frac{1}{t^{1-2\alpha}} \int_{t_0}^t s^{-\alpha} dZ(s).$$

After some manipulations, the above expression can be rewritten as

$$h(t) = A(t) - B(t), \quad (74)$$

where

$$A(t) = \frac{1}{t^{1-2\alpha}} \frac{1}{2} \frac{1}{1 - 2\alpha} \left( t^{1-2\alpha} - t_0^{1-2\alpha} \right),$$

$$B(t) = \frac{1}{t^{1-2\alpha}} \int_{t_0}^t s^{-\alpha} dZ(s).$$

Thus, in the case of $\alpha < 1/2$, we have

$$\lim_{t \to \infty} A(t) = \frac{1}{2} \frac{1}{1 - 2\alpha}. \quad (75)$$

Moreover, $B(t)$ has a mean of 0 and variance of

$$\frac{1}{t^{2(1-2\alpha)}} \frac{1}{1 - 2\alpha} \left( t^{1-2\alpha} - t_0^{1-2\alpha} \right),$$

which converges to 0. We then obtain

$$\lim_{t \to \infty} B(t) = 0 \quad a.s. \quad (76)$$

Therefore, equations (73)–(76) imply that

$$\lim_{t \to \infty} \frac{c_2(t)}{c_1(t)} = 0 \quad a.s. \quad (77)$$

That is, investor 2 cannot survive even though his beliefs converge to the truth because he learns too slowly!
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Table 1: The Impact of the Time Discount Rate on Investors’ Survival

| $|\delta/\kappa|$ | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| $\rho_1 - \rho_2$ | 0.0024 | 0.0096 | 0.0216 | 0.0384 |

This table demonstrates how the time discount rate determines investors’ survival. More precisely, investor 1 has rational expectations and investor 2 has a degree of irrationality of $\delta$, as expressed in equation (6). The Sharpe ratio $\kappa$ is set to be 0.277, according to the estimate based on U.S. data from 1891–1998 in Campbell (2003). The first row reports investor 2’s mis-perception of the Sharpe ratio. For example, $|\delta/\kappa| = 0.25$ means that investor 2 over- or underestimates the Sharpe ratio by 25%. The second row reports the minimal difference in the time discount rates, $\rho_1 - \rho_2$, for investor 2 to drive investor 1 out of the market. For example, the first column 0.0024 means that even if investor 2 over- or underestimates the Sharpe ratio by 25%, he can still dominate the market if $\rho_1 - \rho_2 > 0.0024$.

Table 2: The Impact of the Utility Function on Investors’ Survival

| $|\delta/\kappa|$ | 0.25 | 0.5 | 0.75 | 1 |
|---|---|---|---|---|
| $\gamma_1 - \gamma_2$ | 0.13 | 0.54 | 1.21 | 2.14 |

This table demonstrates how the utility function determines investors’ survival. More precisely, investor 1 has rational expectations and investor 2 has a degree of irrationality of $\delta$, as expressed in equation (6). The Sharpe ratio $\kappa$ and the mean economy growth rate $g_D$ are are set to be 0.277 and 1.789%, respectively, according to the estimate based on U.S. data from 1891–1998 in Campbell (2003). The first row reports investor 2’s mis-perception of the Sharpe ratio. For example, $|\delta/\kappa| = 0.25$ means that investor 2 over- or underestimates the Sharpe ratio by 25%. The second row reports the minimal difference in the utility functions, $\gamma_1 - \gamma_2$, for investor 2 to drive investor 1 out of the market. For example, the first column 0.13 means that even if investor 2 over- or underestimates the Sharpe ratio by 25%, he can still dominate the market if $\gamma_1 - \gamma_2 > 0.13$. 
At time 0, the economy is populated by investors 1 and 2 and each of them has a half share of the stock. Investor 1 has rational expectations and investor 2’s degree of irrationality is $\delta$, as expressed in equation (6). The economy is simulated 100,000 times. For each simulation path, the equilibrium at time $T = 100$ years is computed. The following plots report the simulated probability density functions of investor 2’s consumption share, $c_2(T)/(c_1(T) + c_2(T))$, wealth share, $W_2(T)/(W_1(T) + W_2(T))$, impacts on the stock price and interest rate, $S(T)/S^*(T)$, $r(T)/r^*(T)$, where $S^*(T)$ and $r^*(T)$ are the stock price and interest rate in the economy populated by investor 1 only. Panels A–D are for the cases of $\delta = 0.25 \kappa$, $\delta = 0.5 \kappa$, $\delta = 0.75 \kappa$, and $\delta = \kappa$ respectively. The time discount rate $\rho$ is set to be 0.01. The parameter values for $g_D$, $\sigma_D$ and $\kappa$ are from the estimate based on U.S. data from 1891–1998 in Campbell (2003): $g_D = 1.789\%$, $\sigma_D = 3.218\%$, and $\kappa = 0.277$.

Panel A: $\delta = 0.25 \kappa$
Figure 1: Investor 2’s Impacts after 100 Years (con’t).

Panel B: $\delta = 0.5\kappa$
Figure 1: Investor 2’s Impacts after 100 Years (con’t).

Panel C: $\delta = 0.75\kappa$
Figure 1: Investor 2’s Impacts after 100 Years (con’t).

Panel D: $\delta = \kappa$

Investor 2’s Consumption Share ($T=100$)

Investor 2’s Wealth Share ($T=100$)

Investor 2’s Impact on Stock Price ($T=100$)

Investor 2’s Impact on Interest Rate ($T=100$)
Figure 2: Investor 2’s Expected Wealth Share.

At time 0, the economy is populated by investors 1 and 2 and each of them has a half share of the stock. Investor 1 has rational expectations and investor 2’s degree of irrationality is $\delta$, as expressed in equation (6). The economy is simulated 1000 times. For each simulation path, investor 2’s wealth shares at time $t = 50, 100, 150, \ldots, 1000$ years are computed. For each time $t$, the mean and standard deviation of investor 2’s wealth share across the 1000 simulation paths are computed. All the standard deviations are smaller than 1% and so the following figures only plot the mean of investor 2’s wealth share against time $t$. Panels A–D are for the cases of $\delta = 0.25\kappa$, $\delta = 0.5\kappa$, $\delta = 0.75\kappa$, and $\delta = \kappa$ respectively. The time discount rate $\rho$ is set to be 0.01 and the parameter values for $g_D$, $\sigma_D$ and $\kappa$ are from the estimate based on U.S. data from 1891–1998 in Campbell (2003): $g_D = 1.789\%$, $\sigma_D = 3.218\%$, and $\kappa = 0.277$. 

![Panel A: $\delta = 0.25\kappa$](image1)

![Panel B: $\delta = 0.5\kappa$](image2)

![Panel C: $\delta = 0.75\kappa$](image3)

![Panel D: $\delta = \kappa$](image4)