

ADDING AND SUBTRACTING BLACK-SCHOLES: A NEW APPROACH TO APPROXIMATING DERIVATIVE PRICES IN CONTINUOUS-TIME MODELS*

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Abstract

We develop a new approach to approximating asset prices in the context of multifactor continuous-time models. For any pricing model that lacks a closed-form solution, we provide a solution, which relies on the approximation of the intractable model through a known, “auxiliary” one. We derive an expression for the difference between the true (but unknown) price and the auxiliary one, which we approximate in closed-form, and use to create increasingly improved refinements to the initial mispricing induced by the auxiliary model. The approach is intuitive, simple to implement and leads to fast and extremely accurate approximations. We illustrate this method in a variety of contexts, including option pricing with stochastic volatility, volatility contracts and the term-structure of interest rates.

KEYWORDS: Asset pricing; stochastic volatility; closed-form approximations.

JEL-CLASSIFICATION: G12, G13.

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1 Introduction

The last decade has witnessed an ever increasing demand for new models addressing a number of empirical puzzles in financial economics, which relate to pricing, hedging, and spanning derivatives contracts (e.g., Bakshi and Madan, 2000; Duffie, Pan and Singleton, 2000), the term structure of interest rates (e.g., Ahn, Dittmar and Gallant, 2002; Dai and Singleton, 2002), or the aggregate stock market (e.g., Gabaix, 2008; Menzly, Santos and Veronesi, 2004). The vast majority of these models rely on a continuous-time framework, which is by now one of the most celebrated analytical tools in financial economics. Market practitioners have also increasingly relied on continuous-time models (e.g., Brigo and Mercurio, 2006). The reason for this consensus about the benefits of continuous-time modeling is that within this framework, we are able to provide elegant representations for the price of a variety of contingent claims. At the same time, continuous-time models call for an old and well-known practical issue: how do we go about dealing with models not solved in closed-form?

As is well-known, closed-form solutions for asset prices constitute the exception, rather than the norm. This fact has led financial economists and practitioners to single out classes of models for which a solution could indeed be found, as in the celebrated affine class (Duffie, Pan and Singleton, 2000; Heston, 1993), considered to be the benchmark, and in other classes including quadratic models (Ahn, Dittmar and Gallant, 2002) or linearity-generating processes (Gabaix, 2008, 2009).

However, it is an open question as to whether the assumptions introduced by these models clash with the actual empirical behavior of the state variables in the economy. Quite often, models with closed-form solutions rest on simplifying assumptions that are typically untested, for the sake of analytical tractability. This circumstance might be problematic, once we move towards a quantitative assessment of these models: how would we know whether, say, the reason for a model's rejection would lie in its very economic rationale or, rather, the mere simplifying assumptions underlying it? The role of simplifying assumptions has also been called into question during the 2007 subprime crisis, which clearly revealed how a small change in the assumptions underlying a model can then have dramatic effects on the ultimate pricing of derivative products (see IMF, 2008).

In dealing with models not solved in closed-form, we typically rely on two alternative approaches. The first approach hinges upon the numerical solution to a partial differential equation obtained through, say, finite-difference or Fourier-inversion methods (Schwarz, 1978; Hull and White, 1990; Scott, 1997). The second approach, initiated by Boyle (1977), relies on Monte Carlo simulations, in which a large number of trajectories needs to be generated for the state variables underlying the asset pricing model. Both methods can be cumbersome to implement and, computationally, quite time-consuming.

This paper develops a new conceptual framework to compute asset prices in nonlinear, multifactor diffusion settings. We develop closed-form approximations to any given contingent claim model, which are easy to implement and require very little computer power. Our main idea is to choose an “auxiliary” pricing model for which a solution is available in closed-form. For example, we can choose affine models to be the auxiliary models, as we shall illustrate throughout the

whole paper. Additional examples of candidate auxiliary models are the quadratic models studied by Ahn, Dittmar and Gallant (2002) and the linearity-generating processes introduced by Gabaix (2008). For any auxiliary model, we derive an expression for the difference between the unknown price of the model of interest and the auxiliary one. This expression takes the form of a conditional expectation taken under the risk-neutral probability, which, under regularity conditions, can be cast in terms of a Taylor series expansion. We approximate the unknown price, by retaining a finite number of terms from this series. Our method is highly general and therefore applicable in a wide range of settings, which range from the pricing of options and the computation of the associated Greeks, to the pricing of bonds and variance swaps. We develop several examples to illustrate how to use our general insights, and provide numerical results that show that our method is quite precise and easily implemented.

Our method relies, as explained, on Taylor series expansions of conditional expectations. Similar expansions are widely used in financial econometrics and empirical finance (see, e.g., Aït-Sahalia, 2002; Aït-Sahalia and Yu, 2006; Aït-Sahalia and Kimmel, 2009; Bakshi, Ju and Ou-Yang; 2006; Schaumburg, 2004). A key feature in this literature is the expansion of a conditional expectation of a continuous-time variable, say some conditional moment related to the short-term interest rate expected to prevail over a small time-span - e.g., one day or one week at most. Such “small time expansions” are less useful, when the objective is to approximate option pricing models, either because (i) the presence of optionality leads to payoff functions that are not differentiable, as for example, in the simple European option pricing case, or because (ii) the maturity of the derivative contracts might occur at long maturity dates, as for example, in the term structure of interest rates. For these reasons, small time expansions have not been applied to asset pricing models previously,¹ although they are reconsidered in recent work by Kimmel (2008), which we discuss in a moment.

Our approach still relies on series expansions of conditional expectations, but works differently. Rather than being applied directly to payoff functions, our expansions apply to pricing errors that summarize the mispricing between the true pricing function and the auxiliary pricing function we choose to approximate the true model by. These pricing errors are typically differentiable even if the payoffs are not. In fact, after completing this paper, we came across the work of Kimmel (2008), who develops a clever method to deal with expansions of the payoff functions, which can be used to address the issues related to long maturity dates. Although Kimmel’s method can not be applied to deal with payoffs that are not differentiable, it can be used in efficient conjunction with ours, to implement closed-form approximations to our pricing errors, which, as noted, are typically differentiable.

The method introduced in this article can be also interpreted, as an expansion of the risk-neutral probability implied by the model of interest, around that of some auxiliary model chosen by the user. As such, our approach shares similarities with the strand of literature where option prices are computed through an approximation of the risk-neutral density underlying the true pricing

¹One early and isolated exception appears in Chapman, Long and Pearson (1999), which is indeed a special case of our method, as we shall explain. However, this special case does not allow one to deal with derivatives written on non-differentiable payoffs.

model, as in Abadir and Rockinger (2003) or in the “saddlepoint approximations” considered by Aït-Sahalia and Yu (2006), Rogers and Zane (1999), or Xiong, Wong and Salopek (2005). In fact, approximating the risk-neutral probability is a special case of our approach, as we shall explain. Our method carries some advantages over approximations of conditional densities, when applied to asset pricing. First, because it relies on a direct expansion of asset prices, our method avoids the numerical computation of multidimensional Riemann integrals against an approximate conditional density. This feature is attractive in multifactor models such as those that involve stochastic interest rates, stochastic volatility or macro-finance determinants of the yield curve. Second, the expansion we provide, carries new and interesting economic content, as we shall illustrate. For example, we shall see that approximating stochastic volatility models through our approach leads to errors, which we can interpret as hedging costs arising through the use of misspecified Black-Scholes deltas. Finally, we provide an explicit expression for the difference between the pricing function of the true and the auxiliary model, which leads to a more direct analysis of the pricing error and simpler approximations.

The paper is organized as follows. In the next section, we illustrate our methods through three asset pricing examples. In Section 3, we develop a general framework to approximate asset prices, provide extensions that allow for the computation of sensitivities of derivative prices and, finally, relate our approach to those relying on the expansion of risk-neutral probabilities. In Section 4, we assess the numerical performance of our methods in concrete applications including the yield curve and option pricing with stochastic volatility. Section 5 concludes. A technical appendix provides details omitted from the main text.

2 The Approximation Method in Three Examples

We illustrate the basic ideas underlying our method by working out three examples, ranked in order of increasing complexity: (i) the pricing of variance contracts, (ii) the pricing of European options within the generalized Black-Scholes model, and (iii) the pricing of bonds in single-factor models of the short-term rate.

2.1 Log-contracts and Variance Swaps

Our basic example pertains to the recent financial innovation related to variance swaps, which are contracts guaranteeing a payoff linked to the realization of the future variance of some asset price. As is well-known, the forward price of any liquid asset, $F(t)$ say, is a martingale under the risk-neutral probability. Moreover, suppose that $F(t)$ exhibits stochastic volatility, as follows:

$$\frac{dF(t)}{F(t)} = \sigma(t) dW(t),$$

where $W(t)$ is a Brownian motion under the risk-neutral probability, and the instantaneous variance follows a continuous-time ARCH process (Nelson, 1990),

$$d\sigma^2(t) = \kappa(\alpha - \sigma^2(t))dt + \xi\sigma^2(t)dW_\sigma(t), \quad (1)$$

for some positive constants κ , α and ξ , and a Brownian motion $W_\sigma(t)$ defined under the risk-neutral probability.

By entering into a variance swap at time t , the holder of the contract will receive, at some time T , a payoff proportional to, $\int_t^T \sigma^2(u)du - \sigma_{\text{strike}}^2$, for some constant σ_{strike}^2 . Typically, the variance strike, σ_{strike}^2 , is set so as to make the contract worthless at the time of origination, t , consistently with the market practice related the more familiar interest rate swaps. Then, if the short-term rate is independent of the forward price variance, it must be that in the absence of arbitrage opportunities, the variance strike equals the expected future integrated variance, viz

$$\sigma_{\text{strike}}^2 = \int_t^T \mathbb{E}_{x,t}[\sigma^2(u)]du = -2\mathbb{E}_{x,t}\left(\log \frac{F(T)}{F(t)}\right),$$

where $\mathbb{E}_{x,t}[\sigma^2(u)] = \mathbb{E}[\sigma^2(u) | \sigma^2(t) = x]$ denotes the conditional mean, and the last equality follows by a simple application of Itô's lemma. The last term is the payoff of the so-called log-contract introduced by Neuberger (1994), which, as shown in many papers (Bakshi and Madan, 2000; Britten-Jones and Neuberger, 2000; Carr and Madan, 2001; Demeterfi, Derman, Kamal and Zou, 1999), is equal to a certain portfolio of options, which is now being used by the CBOE to compute the new VIX index. Alternatively, one may use the parametric model in Eq. (1) to find σ_{strike}^2 and, hence, price the contract. Then, one can calibrate the parameters κ and α to make σ_{strike}^2 consistent with the VIX index, and proceed to use the model in Eq. (1) to price other derivative assets. Needless to say, to perform these tasks, it is crucial to compute the expectation of the future variance, $\mathbb{E}_{x,t}[\sigma^2(u)]$.

In the context of the model in Eq. (1), it is well-known that $\mathbb{E}_{x,t}[\sigma^2(u)]$ has a closed-form expression, which leads to a closed-form solution for the variance strike as of any time t , denoted as: $w(x, t) = \int_t^T \mathbb{E}_{x,t}[\sigma^2(u)]du$. For the sake of this introductory example, suppose we are unaware of this solution, and wish to approximate the variance strike for the model in Eq. (1) with another variance strike that we can compute. Consider, for instance, the following variant of the Hull and White (1987) model, in which the instantaneous variance is a martingale under the risk-neutral probability,

$$d\sigma^2(t) = \xi\sigma^2(t)dW_\sigma(t).$$

For this model, $\mathbb{E}_{x,t}^0[\sigma^2(u)] = x$, where $\mathbb{E}_{x,t}^0$ denotes the conditional expectation under the Hull and White (1987) model, and the variance strike is just $w_0(x, t) \equiv x(T - t)$.

We now illustrate how to use the ‘‘auxiliary’’ variance strike predicted by the Hull and White model, $w_0(x, t)$, to approximate the supposedly unknown variance strike, $w(x, t)$. First, note that the variance strike is the undiscounted sum of all its future ‘‘dividends,’’ where each of these

dividends, say that as of time $t+u$, is the instantaneous variance at time $t+u$. Therefore, the sum of the expected instantaneous capital gain and the instantaneous dividend (the variance) must be zero under the risk-neutral probability, in the absence of arbitrage, $\frac{d}{du} \mathbb{E}_{x,t}[w(\sigma^2(t+u), t+u)]|_{u=0} + x = 0$. That is, the variance strike solves the following partial differential equation,

$$0 = Lw(x, t) + x, \quad (2)$$

with boundary condition $w(x, T) = 0$, where L denotes the infinitesimal generator of $\sigma^2(t)$:

$$Lw(x, t) = \frac{\partial w(x, t)}{\partial t} + \kappa(\alpha - x) \frac{\partial w(x, t)}{\partial x} + \frac{1}{2} \xi^2 x^2 \frac{\partial^2 w(x, t)}{\partial x^2}.$$

Likewise, $w_0(x, t)$, the variance strike predicted by the auxiliary Hull and White model, satisfies,

$$0 = \frac{\partial w_0(x, t)}{\partial t} + x, \quad (3)$$

with boundary condition $w_0(x, T) = 0$.

Our key idea, now, is to subtract Eq. (3) from Eq. (2). A simple computation shows that the mispricing arising from the use of the wrong model, $\Delta w(x, t) \equiv w(x, t) - w_0(x, t)$, satisfies,

$$0 = L\Delta w(x, t) + (T - t) \delta(x),$$

with boundary condition $\Delta w(x, T) = 0$, and “mispricing function” δ given by: $\delta(x) = \kappa(\alpha - x)$.

The solution to the previous equation, provided it exists, can be represented as a conditional expectation, due to the well-known Feynman-Kac theorem (see, e.g., Karatzas and Shreve, 1991). The result is the following representation of the variance mispricing,

$$\Delta w(x, t) = \int_t^T (T - s) \mathbb{E}_{x,t}[\delta(x(s))] ds. \quad (4)$$

The expectation inside the integral can be written explicitly, in terms of the infinitesimal generator associated with the model of interest (1), L , as follows,

$$\mathbb{E}_{x,t}[\delta(x(s))] = \sum_{n=0}^{\infty} \frac{(s-t)^n}{n!} L^n \delta(x), \quad (5)$$

where, by a direct computation, $L^n \delta(x) = \kappa(-\kappa)^n (\alpha - x)$.

Our approximation to the variance mispricing, $\Delta w(x, t)$, is obtained by replacing the expectation $\mathbb{E}_{x,t}[\delta(x(s))]$ in Eq. (4) with only the first N terms of the series expansion in Eq. (5), as follows:

$$\Delta w_N(x, t) = \int_t^T (T - s) \sum_{n=0}^N \frac{(s-t)^n}{n!} L^n \delta(x) ds. \quad (6)$$

Accordingly, our approximation to the variance strike $w(x, t)$ is:

$$w_N(x, t) = w_0(x, t) + \Delta w_N(x, t) = x(T - t) + \kappa(\alpha - x) \sum_{n=0}^N \frac{(T - t)^{n+2}}{(n + 2)!} (-\kappa)^n,$$

where the second equality follows by the evaluation of the integral in Eq. (6). It is easily checked that as N increases, $w_N(x, t)$ approaches the true variance strike, $w(x, t) = \alpha(T - t) + (x - \alpha)(1 - e^{-\kappa(T-t)})/\kappa$.

2.2 The Generalized Black-Scholes Option Pricing Model

Next, we illustrate our basic ideas in the context of the pricing of options written on traded assets. Suppose that the price of a stock, $S(t)$ say, is the solution to

$$\frac{dS(t)}{S(t)} = rdt + \sigma(S(t), t) dW(t), \quad (7)$$

where $W(t)$ is a standard Brownian motion under the risk-neutral probability, and r is the short-term rate, a constant. A European call option pays $b(S(T)) \equiv \max\{S(T) - K, 0\}$ at maturity time $T > 0$, where $K > 0$ is the strike price. Let $w(x, t)$ be the price of the option at time $t < T$ given the current stock price is $S(t) = x$.

Let L be the infinitesimal generator associated with Eq. (7),

$$Lw(x, t) = \frac{\partial w(x, t)}{\partial t} + rx \frac{\partial w(x, t)}{\partial x} + \frac{1}{2} \sigma^2(x, t) x^2 \frac{\partial^2 w(x, t)}{\partial x^2}. \quad (8)$$

Under standard regularity conditions on the volatility function $\sigma(x, t)$, the option price, $w(x, t)$, satisfies:

$$0 = Lw(x, t) - rw(x, t), \quad (9)$$

with boundary condition $w(x, T) = b(x)$, for all x .

In general, the solution to Eq. (9), provided it exists, is not known in closed-form. The starting point of our approximation method is, as in the previous example on variance swaps, the choice of an auxiliary model that can be solved in closed-form. In this context, the Black and Scholes (1973) (BS, henceforth) model is a natural candidate. For this model, the volatility in Eq. (7) is a constant, i.e. $\sigma(x, t) \equiv \sigma_0$, for all x, t . Accordingly, the BS option price, $w^{\text{bs}}(x, t; \sigma_0)$, say, is solution to,

$$0 = L_0 w^{\text{bs}}(x, t; \sigma_0) - r w^{\text{bs}}(x, t; \sigma_0), \quad (10)$$

where $w^{\text{bs}}(x, T; \sigma_0) = \max\{x - K, 0\}$, and the associated infinitesimal operator, L_0 , is the same as in Eq. (8), but with σ_0 replacing $\sigma(x, t)$.

Proceeding as we did in Section 2.1, we now subtract Eq. (10) from Eq. (9). The result is that

the price difference, $\Delta w(x, t; \sigma_0) \equiv w(x, t) - w^{\text{bs}}(x, t; \sigma_0)$, satisfies,

$$0 = L\Delta w(x, t; \sigma_0) - r\Delta w(x, t; \sigma_0) + \delta(x, t; \sigma_0), \quad (11)$$

with boundary condition $\Delta w(x, T; \sigma_0) = 0$ for all x , where our mispricing function is now:

$$\delta(x, t; \sigma_0) \equiv \frac{1}{2} (\sigma(x, t)^2 - \sigma_0^2) x^2 \frac{\partial^2}{\partial x^2} w^{\text{bs}}(x, t; \sigma_0). \quad (12)$$

Since $w^{\text{bs}}(x, t; \sigma_0)$ is known, we can compute $\delta(x, t; \sigma_0)$. By relying on the Feynman-Kac representation of the solution to Eq. (11), the unknown pricing function can be expressed as the sum of the Black-Scholes price plus a conditional moment, which we shall interpret in a moment:

$$w(x, t) = w^{\text{bs}}(x, t; \sigma_0) + \mathbb{E}_{x,t} \left[\int_t^T e^{-r(u-t)} \delta(S(u), u; \sigma_0) du \right]. \quad (13)$$

The interpretation of the mispricing function δ in Eq. (12) relates to the hedging cost arising while evaluating and hedging the option through the BS formula. Precisely, suppose a trader sells the option and wishes to hedge against it through a self-financing strategy, in which he trades the underlying stock using the BS delta, $\partial w^{\text{bs}}(x, t; \sigma_0) / \partial x$. Then, as shown by El Karoui, Jeanblanc-Picqué and Shreve (1998), and further elaborated by Corielli (2006), our function δ in Eq. (12) is interpreted as the instantaneous increment in the total hedging cost arising from the use of a wrong model (the BS model) to hedge against the true model in Eq. (7).

The conditional moment in Eq. (13) is taken under the stock price dynamics given by Eq. (7). Therefore, it is in general impossible to obtain a closed-form expression for the second term in Eq. (13). To make this formula operational, we make use of a series expansion of the conditional moment in Eq. (13) in terms of the corresponding infinitesimal generator. As shown in the Appendix (see Proposition A.3), Eq. (13) is indeed equivalent to:

$$w(x, t) = w^{\text{bs}}(x, t; \sigma_0) + \sum_{n=0}^{\infty} \frac{(T-t)^{n+1}}{(n+1)!} \bar{L}^n \delta(x, t; \sigma_0), \quad (14)$$

where $\bar{L}\delta = L\delta - r\delta$. In practice, this formula needs to be truncated, yielding:

$$w_N(x, t; \sigma_0) \equiv w^{\text{bs}}(x, t; \sigma_0) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} \bar{L}^n \delta(x, t; \sigma_0), \quad (15)$$

for some $N \geq 0$. For example, a first order approximation ($N = 0$) is given by $w_0(x, t; \sigma_0) \equiv w^{\text{bs}}(x, t; \sigma_0) + (T-t)\delta(x, t; \sigma_0)$. Naturally, the unknown option price w in Eq. (14) does not depend on σ_0 , although its “truncation” w_N does. In Section 4.1, we discuss choices of the nuisance parameter, σ_0 . In our numerical experiments reported in Section 4.1, we find that the numerical accuracy of $w_N(x, t; \sigma_0)$ does not crucially depend on the choice of σ_0 .

2.3 Bond Pricing in a Single-Factor Model

For our third example, we consider the pricing of bonds in a single-factor model of the short-term interest rate. Suppose that the short-term rate r is solution to

$$dr(t) = \mu(r(t), t)dt + \sigma(r(t), t)dW(t), \quad (16)$$

for some drift and diffusion functions $\mu(x, t)$ and $\sigma(x, t)$, and a standard Brownian motion $W(t)$ defined under the risk-neutral probability. Let $w(x, t)$ be the price as of time t of a default-free bond maturing at time $T > t$, when $r(t) = x$. Under standard regularity conditions on $\mu(x, t)$ and $\sigma(x, t)$, $w(x, t)$ is solution to,

$$0 = Lw(x, t) - xw(x, t), \quad Lw = \frac{\partial w}{\partial t} + \mu \frac{\partial w}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 w}{\partial x^2}, \quad (17)$$

with boundary condition $w(x, T) = 1$.

Next, let us introduce, as usual, an auxiliary model,

$$dr(t) = \mu_0(r(t), t)dt + \sigma_0(r(t), t)dW(t),$$

where $\mu_0(x, t)$ and $\sigma_0(x, t)$ are some drift and diffusion functions. Associated with this model is a bond pricing function, $w_0(x, t; \theta_0)$, which solves Eq. (17), with boundary condition $w_0(x, T) = 1$, but with μ_0 and σ_0 replacing μ and σ . In $w_0(x, t; \theta_0)$, the vector θ_0 is notation we use to denote the nuisance parameter vector in the two auxiliary functions $\mu_0(x, t)$ and $\sigma_0(x, t)$. It parallels the BS σ_0 of the previous section.

It is easy to show that the price difference, $\Delta w(x, t; \theta_0) \equiv w(x, t) - w_0(x, t; \theta_0)$, satisfies:

$$0 = L\Delta w(x, t; \theta_0) - x\Delta w(x, t; \theta_0) + \delta(x, t; \theta_0), \quad (18)$$

where $\Delta w(x, T) = 0$, and the mispricing function is,

$$\delta(x, t; \theta_0) = (\mu(x, t) - \mu_0(x, t)) \frac{\partial w_0(x, t)}{\partial x} + \frac{1}{2} (\sigma^2(x, t) - \sigma_0^2(x, t)) \frac{\partial^2 w_0(x, t)}{\partial x^2}. \quad (19)$$

Note, the function summarizing the mispricing arising from the use of the auxiliary model, $\delta(x, t; \theta_0)$, has now a more complex structure than that we find in Section 2.2 in the option pricing case. Its second component, the convexity adjustment, is now familiar, by the results in Section 2.2. Its first term, which is new, arises because the short-term rate is obviously not a traded risk, which makes the two drifts under the risk-neutral probability, μ and μ_0 , differ. In the option example dealt with in Section 2.2, instead, the asset underlying the contract is tradable, and is expected to appreciate at an instantaneous rate of $r dt$, under the risk-neutral probability, independently of the evaluation model. A choice that simplifies the function δ in Eq. (19) is $\mu_0 = \mu$, which we shall use in our numerical experiments of Section 4.2.

By the Feynman-Kac representation, the solution to Eq. (18) is,

$$\Delta w(x, t; \theta_0) = \int_t^T \mathbb{E}_{x,t} \left[\exp \left(- \int_t^u r(s) ds \right) \delta(r(u), u; \theta_0) \right] du.$$

Using the same type of power series expansions as in Sections 2.1 and 2.2, we obtain the following approximating formula for the bond price function $w(x, t)$:

$$w_N(x, t; \theta_0) = w_0(x, t; \theta_0) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} \bar{L}^n \delta(x, t; \theta_0), \quad (20)$$

where now \bar{L} is defined as $\bar{L}\delta(x, t; \theta_0) = L\delta(x, t; \theta_0) - x\delta(x, t; \theta_0)$.

As for the option pricing model discussed in Section 2.2, the approximating bond price, $w_N(x, t)$, depends on some nuisance parameters, which arise through the use of the auxiliary model. For example, assume that the drift of the auxiliary model is chosen so as to match the drift of the model we approximate, $\mu = \mu_0$. Suppose, also, that the diffusion function of the auxiliary model is a constant. This constant, then, is the nuisance parameter. Section 4.2 provides examples based on this case of a constant nuisance parameter, and discusses simple choices of it, which still lead to quite reliable approximation outcomes.

3 A General Approximating Pricing Formula

In this section, we derive a general approximation formula for asset prices in models not solved in closed-form, obtained following the same lead as that for the three examples discussed in Section 2. In Section 3.1, we introduce notation for the model we approximate and its auxiliary counterpart, and provide our approximating formula. In Section 3.2, we discuss approximations for derivatives of the pricing functions of interest, which can be useful for hedging purposes. In Section 3.3, we explain how our approach relates to methods that rely on the expansion of risk-neutral densities.

3.1 The model and its approximation

We consider a multifactor model in which a d -dimensional vector of state variables $x(t)$ affects all asset prices in the economy. We assume that under the risk-neutral probability $x(t)$ satisfies:

$$dx(t) = \mu(x(t), t) dt + \sigma(x(t), t) dW(t), \quad (21)$$

where $W(t)$ is a d -dimensional standard Brownian motion under the risk-neutral probability, and $\mu(x, t)$ and $\sigma(x, t)$ are some drift and diffusion functions. Let $w(x, t)$ be the price of a derivative written on the realization of $x(T)$, for some $T > t$, when the current state is $x(t) = x$. The derivative price is characterized by three components: First, its payoff function at T as given by $b(x(T))$, for some function $b(x)$. Second, let $R(x, t)$ denote the instantaneous short-term interest rate at time t , when the state vector is $x(t) = x$. Finally, let $c(x, t)$ be the instantaneous coupon

rate promised by the asset at time t .

Define the infinitesimal generator operator L associated to Eq. (21),

$$Lw(x, t) = \frac{\partial w(x, t)}{\partial t} + \sum_{i=1}^d \mu_i(x, t) \frac{\partial w(x, t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \sigma_{ij}^2(x, t) \frac{\partial^2 w(x, t)}{\partial x_i \partial x_j}. \quad (22)$$

The derivative price, $w(x, t)$, is then the solution to the following partial differential equation:

$$Lw(x, t) + c(x, t) = R(x, t)w(x, t), \quad (23)$$

with boundary condition $w(x, T) = b(x)$ for all x . In words, an investment into this asset must be such that the expected instantaneous capital gain, $Lw(x, t)$, plus the instantaneous coupon rate, $c(x, t)$, must equal the instantaneous yield on a safe asset, under the risk-neutral probability.

To approximate the unknown price $w(x, t)$, we introduce an auxiliary model,

$$dx_0(t) = \mu_0(x_0(t), t) dt + \sigma_0(x_0(t), t) dW(t), \quad (24)$$

for some drift and diffusion functions $\mu_0(x, t)$ and $\sigma_0(x, t)$. Our objective is a suitable expansion of the initial model about such an auxiliary model. We assume that the dimension of the auxiliary model is the same as that of the initial model, i.e. x_0 is a d -dimensional vector. This assumption does not entail any loss of generality, since we can always add constant components, as we now explain. Suppose, for example, that we wish to consider an auxiliary model with a lower dimension, where the state vector $y(t) \in \mathbb{R}^m$, with $m < d$, solves, for some drift and diffusion functions μ_Y and σ_Y :

$$dy(t) = \mu_Y(y(t), t) dt + \sigma_Y(y(t), t) dW_1(t),$$

and $W_1(t)$ is a m -dimensional standard Brownian motion. The vector process $[y^\top \ x_{m+1} \ \cdots \ x_d]^\top$, where the last $d - m$ components remain constant over time, is then a solution to Eq. (24) with:

$$\mu_{0,i}(x, t) = \begin{cases} \mu_{Y,i}(y, t), & 1 \leq i \leq m \\ 0, & \text{otherwise} \end{cases} \quad \sigma_{0,ij}(x, t) = \begin{cases} \sigma_{Y,ij}(y, t), & 1 \leq i, j \leq m \\ 0, & \text{otherwise} \end{cases}$$

In Section 4.3, we use this modeling trick to approximate the price of options in models with stochastic volatility, using the Black-Scholes model as an auxiliary device.

As for the derivative associated with the auxiliary market, we assume that the derivative is worth $b_0(x(T))$ at time T , for some function $b_0(\cdot)$. This complication helps illustrate a few properties of our approximation methods arising within the pricing of bonds, as we shall explain in Section 4.2. However, in most cases, one will choose $b_0(x) = b(x)$ such that the auxiliary pricing function, $w_0(x, t)$, mimics $w(x, t)$. Finally, and crucially, we assume that we have a closed-form solution $w_0(x, t)$ for the pricing function in the markets where the state vector satisfies Eq. (24). To save on notation, we do not make explicit that the pricing function $w_0(x, t)$ depends on nuisance parameters, as we did in our introductory examples of the previous section.

The price difference, $\Delta w(x, t) \equiv w(x, t) - w_0(x, t)$, satisfies,

$$L\Delta w(x, t) + \delta(x, t) = R(x, t) \Delta w(x, t), \quad (25)$$

with boundary condition $\Delta w(x, T) = d(x)$. The two adjustment terms are given by

$$d(x) = b(x) - b_0(x), \quad (26)$$

and

$$\delta(x, t) = \sum_{i=1}^d \Delta\mu_i(x, t) \frac{\partial w_0(x, t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \Delta\sigma_{ij}^2(x, t) \frac{\partial^2 w_0(x, t)}{\partial x_i \partial x_j}, \quad (27)$$

where

$$\Delta\mu_i(x, t) = \mu_i(x, t) - \mu_{0,i}(x, t), \quad \Delta\sigma_{ij}^2(x, t) = \sigma_{ij}^2(x, t) - \sigma_{0,ij}^2(x, t).$$

Under standard regularity conditions reviewed in the Appendix, we can apply the Feynman-Kac representation of the solution to the derivative mispricing in Eq. (25), $\Delta w(x, t)$, to obtain the following representation of the asset price, $w(x, t)$, in terms of that arising within the auxiliary model, $w_0(x, t)$:

Theorem 1 (Asset Price Representation) *Assume that the two solutions, $w(x, t)$ and $w_0(x, t)$ to Eq. (23) and (25) exist. Then the following identity holds:*

$$\begin{aligned} w(x, t) = & w_0(x, t) + \mathbb{E}_{x,t} \left[\exp \left(- \int_t^T R(x(s), s) ds \right) d(x(T)) \right] \\ & + \int_t^T \mathbb{E}_{x,t} \left[\exp \left(- \int_t^s R(x(u), u) du \right) \delta(x(s), s) \right] ds, \end{aligned} \quad (28)$$

where $x(t)$ satisfies Eq. (21), and d, δ are given in Eq. (26)-(27).

The above representation formula holds under quite weak assumptions.² The right hand side delivers an exact expression for the error arising from the use of the auxiliary model to price the claim, instead of the true model. This representation is useful in its own right, as it shows precisely how the pricing error is related to the auxiliary model.

Yet our main goal is to look for an approximation of the error term in order to adjust the price $w_0(x, t)$ for the error involved. Accordingly, our next step is to approximate the two expectations on the right hand side of Eq. (28) using a series expansion. For the series expansion to hold, we have to impose stronger assumptions. For an N -th order approximation to be valid, we need to assume that $d(x, t)$ and $\delta(x, t)$ are $2N$ times differentiable with respect to x and N times differentiable with respect to t . Under this assumption, we consider the following definition:

²The only condition that could not possibly hold in standard asset pricing models is the linear growth condition that we impose on the drift and diffusion terms. However, this condition is only needed to ensure that the solutions $x(t)$ and $x_0(t)$ to the two stochastic differential equations exist. Other conditions other than the linear growth conditions can be used to ensure these solutions do actually exist.

Definition 1 (Asset Price Approximation) *The N -th order approximation $w_N(x, t)$ to the unknown price $w(x, t)$ in Eq. (28), at time t and state x , is given by:*

$$w_N(x, t) = w_0(x, t) + \sum_{n=0}^N \frac{(T-t)^n}{n!} d_n(x, t) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x, t), \quad (29)$$

where $d_0(x, t) = d(x)$, $\delta_0(x, t) = \delta(x, t)$ and

$$d_n(x, t) = Ld_{n-1}(x, t) - R(x, t)d_{n-1}(x, t), \quad \delta_n(x, t) = L\delta_{n-1}(x, t) - R(x, t)\delta_{n-1}(x, t).$$

In Appendix A, we provide additional regularity conditions under which our asset price approximation formula is valid, asymptotically, in that $w_N(x, t) \rightarrow w(x, t)$ as $N \rightarrow \infty$. It also provides error bounds applying to any fixed approximation order, $N \geq 1$.

Note, finally, that the approximation in Definition 1 is only a means to estimate the right hand side of Eq. (28) in Theorem 1. Other methods might be available. For example, one could approximate the two conditional expectations appearing in the right hand side of Eq. (28) through simulations. However, one might then just use simulations, and directly compute the conditional expectation appearing in the Feynman-Kac representation of $w(x, t)$. The attractive feature of the power expansion in Eq. (29) is, naturally, that, once implemented, it requires virtually no computation time.

3.2 Approximating Greeks

We outline how our approximation methods can be used to obtain closed-form approximations to the partial derivatives of asset prices, which can be useful to estimate Greeks. The approximation to these partial derivatives are readily obtained indeed, by differentiating the approximating formula in Eq. (29) of Definition 1 with respect to the variables of interest.

The approximation of the k -th order derivative of $w(x, t)$ is given by,

$$\frac{\partial^k w_N(x, t)}{\partial x^k} = \frac{\partial^k w_0(x, t)}{\partial x^k} + \sum_{n=0}^N \frac{(T-t)^n}{n!} d_n^{(k)}(x, t) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} \delta_n^{(k)}(x, t),$$

where

$$d_n^{(k)}(x, t) = \frac{\partial^k d_n(x, t)}{\partial x^k}, \quad \delta_n^{(k)}(x, t) = \frac{\partial^k \delta_n(x, t)}{\partial x^k}.$$

The two sequences, $d_n^{(k)}(x, t)$ and $\delta_n^{(k)}(x, t)$, can be evaluated either numerically (using, say, finite-difference methods) or analytically. For example, to compute the approximation to the first-order derivatives, $k = 1$, we use the following recursion: $d_0^{(1)}(x, t) = \partial d(x) / \partial x$, $\delta_0^{(1)}(x, t) = \partial \delta(x, t) / \partial x$ and,

$$d_n^{(1)}(x, t) = Ld_{n-1}^{(1)}(x, t) - R(x, t)d_{n-1}^{(1)}(x, t) + L^{(1)}d_{n-1}(x, t) - \frac{\partial R(x, t)}{\partial x}d_{n-1}(x, t),$$

$$\delta_n^{(1)}(x, t) = L\delta_{n-1}^{(1)}(x, t) - R(x, t)\delta_{n-1}^{(1)}(x, t) + L^{(1)}\delta_{n-1}(x, t) - \frac{\partial R(x, t)}{\partial x}\delta_{n-1}(x, t),$$

where

$$L^{(1)}\phi(x, t) = \sum_{i=1}^d \frac{\partial \mu_i(x, t)}{\partial x} \frac{\partial \phi(x, t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial \sigma_{ij}^2(x, t)}{\partial x} \frac{\partial^2 \phi(x, t)}{\partial x_i \partial x_j}.$$

In Appendix B, we provide full details about the recursive scheme needed to compute the terms $d_n^{(2)}(x, t)$ and $\delta_n^{(2)}(x, t)$ related to the approximation to the second-order partial derivatives of the asset price.

3.3 Risk-Neutral Probabilities

Asset prices are conditional expectations taken under the risk-neutral probability. Approximating asset prices, then, does necessarily entail approximating risk-neutral probabilities. How do our approximation methods precisely relate to those approximating risk-neutral probabilities? In this section, we link the expansion in Theorem 1 of the price $w(x, t)$ about the auxiliary price $w_0(x, t)$, to the expansion of the risk-neutral probability of the asset pricing model around that of the auxiliary pricing model. In Appendix C, we provide a few more technical details, such as those pertaining to the evaluation of risk-neutral densities based on “saddlepoint approximations,” which we show to be special cases of our approximation methods.

For the purpose of simplifying the presentation, let the instantaneous short-term rate R and the coupon c in Eq. (23) be identically zero, $R(x, t) = c(x, t) \equiv 0$. Then, the two prices, $w(x, t)$ and $w_0(x, t)$, are simply:

$$w(x, t) = \int_{\mathbb{R}^d} b(y) p(y, T|x, t) dy, \quad w_0(x, t) = \int_{\mathbb{R}^d} b(y) p_0(y, T|x, t) dy,$$

where p and p_0 are the risk-neutral conditional densities underlying the two models: the true, p , and the auxiliary, p_0 . Clearly, we have:

$$w(x, t) = w_0(x, t) + \int_{\mathbb{R}^d} b(y) \Delta p(y, T|x, t) dy \tag{30}$$

where $\Delta p \equiv p - p_0$ is the difference between the two conditional densities, the risk-neutral “transition discrepancy,” using a terminology due to Aït-Sahalia (1996). It is easy to see that the asset price representation in Theorem 1 implies that the following identity holds true:

$$\int_{\mathbb{R}^d} b(y) \Delta p(y, T|x, t) dy = \int_t^T \mathbb{E}_{x,t}[\delta(x(s), s)] ds, \tag{31}$$

where δ is as in Eq. (27) (see Appendix C). Therefore, our expansion of $\int_t^T \mathbb{E}_{x,t}[\delta(x(s), s)] ds$ in Definition 1, is related to a corresponding expansion of the risk-neutral transition discrepancy, Δp . In fact, in Appendix C, we derive an explicit representation of $\Delta p(y, T|x, t)$ in terms of a conditional expectation (see Eq. (C1)), which highlights the fact that the representation and approximations

of w in Theorem 1 and Definition 1 rely on equivalent representations and approximations of the risk-neutral conditional density.

In spite of this equivalence, our methods offer greater flexibility, as they lead to closed-form approximations for pricing errors that are easily implemented. To illustrate, the right hand side of Eq. (31), which is the pricing error arising from the use of an *auxiliary asset price*, can be easily computed through a power series expansion, as that in Definition 1. In contrast, the left hand side of Eq. (31), which is the pricing error arising from the use of an *auxiliary risk-neutral density*, requires the computation of a Riemann integral. This computation can be cumbersome, especially when the dimension of the model, d , is large. Finally, the previous equivalence holds when the short-term interest rate and the coupon are constant. In general, it is unclear as to how to use approximations of risk-neutral probabilities to deal with conditional expectations such as,

$$\mathbb{E}_{x,t} \left[\exp \left(- \int_t^T R(x(s), s) ds \right) b(x(T)) \right].$$

These cases need to be dealt with in many instances, especially those including the pricing of fixed income products, or derivatives in the presence of stochastic interest rates. Our methods, which rely on approximations directly obtained through auxiliary *pricing* functions (not auxiliary *risk-neutral probabilities*), do handle these cases in a quite natural manner.

4 Numerical Accuracy of Approximation

We assess the performance of our asset price approximations in three applications: (i) option pricing in models with CEV volatility, such as that in Section 2.2; (ii) the term-structure of interest rates; (iii) option pricing with stochastic volatility.

4.1 Option Pricing with CEV Volatility

Consider the generalized BS model in Section 2.2, which, as explained, we wish to approximate through our methods, using the BS model as an auxiliary pricing device. The use of an auxiliary model inevitably leads to a nuisance parameter—a parameter that does not affect the unknown price, but does enter the pricing formula for the auxiliary model. In the BS case, the nuisance parameter is the instantaneous volatility σ_0 in Eq. (10). There are several alternatives to deal with this parameter. For example, let $\hat{\sigma}_0$ be some estimate of σ_0 . Then, we may approximate $w(x, t)$ with $w_N(x, t; \hat{\sigma}_0)$. Alternatively, we may consider,

$$\hat{\sigma}_N(x, t) = \arg \min_{\sigma} (w_N(x, t; \sigma) - w_0(x, t; \sigma))^2,$$

where $w_0(x, t; \sigma) \equiv w^{\text{bs}}(x, t; \sigma)$, in terms of the notation in Section 2.2. As a simple example, we have that for $N = 0$, $\hat{\sigma}_0(x, t) = \sigma(x, t)$. Clearly, $\lim_{N \rightarrow \infty} \hat{\sigma}_N(x, t) = \text{IV}(x, t)$, where $\text{IV}(x, t)$ denotes the Black-Scholes implied volatility, defined by $w(x, t) = w^{\text{bs}}(x, t; \text{IV}(x, t))$. For fixed N ,

then, the unknown option price can be approximated by $w_N(x, t; \sigma_N(x, t))$, or more generally, $w_N(x, t; \sigma_M(x, t))$, where $M \leq N$, as we do in the numerical experiments reported below.

To gauge the performance of the approximation, we consider the CEV model, for which $\sigma(x, t) = \sigma_{\text{cev}}x^{\gamma-1}$, where σ_{cev} is constant and $\gamma > 0$. For this model, the option price is known in closed-form (see Schroder, 1989), which allows us to achieve a precise quantitative assessment of the approximation. In Figure 1, we depict the approximation errors resulting from our method, arising for different levels of the asset price, when the parameter values are $\sigma_{\text{cev}} = 10\%$, $\gamma = \frac{1}{2}$, $r = 5\%$ and, finally, the strike price is $K = 100$ and time-to-maturity is three months. The approximating price is obtained as $w_N(x, t; \hat{\sigma}_0(x))$, where $\sigma_0(x) = \sigma_{\text{cev}}x^{\gamma-1}$, which explains why the percentage errors for $N = 0$ and $N = 1$ coincide. More fundamentally, the errors are several orders of magnitude lower than one percentage point, with only a very small number of correction terms ($N = 3$). Figure 2 depicts the errors arising in pricing the option with a larger maturity (one year): our approximation is still quite accurate in this case, even for the more extreme far-in and far-out of the money options.

4.2 The Term Structure of Interest Rates

The framework developed in Section 3 allows us to approximate asset prices through quite general auxiliary models. This section illustrates our methods and analyzes the numerical performance of two auxiliary models in approximating the (supposedly unknown) solution to the Cox, Ingersoll and Ross (1985) (CIR, henceforth) model of the yield curve, where the short-term rate is solution to:

$$dr(t) = \beta(\alpha - r(t))dt + \sigma\sqrt{r(t)}dW(t), \quad (32)$$

where $\alpha > 0$, $\beta > 0$ and $\sigma > 0$ are constants. In terms of Eq. (23), therefore, the short-term rate is $R(r, t) = r$, where r is solution to Eq. (32).

The first auxiliary model we analyze is just one for which: (i) the payoff paid by the bond is zero, rather than one; and (ii) the short-term rate is the same as in the true data generating mechanism, that in Eq. (32). We argue that the approximating formula we shall come up with is that provided by Chapman, Long and Pearson (1999). We study this case in Section 4.2.1.

In Section 4.2.2, we investigate the performance of our methods when the auxiliary model is such that: (i) the payoff of the bond equals the true payoff, one; and (ii) the auxiliary model is the Vasicek (1977) model, where the short-term rate is solution to:

$$dr_0(t) = \beta_0(\alpha_0 - r_0(t))dt + \sigma_0dW(t), \quad (33)$$

for three constants α_0 , β_0 and σ_0^2 .

4.2.1 A Simple Power Expansion

We consider a quite straight forward auxiliary market, one where drift and diffusion terms coincide with the drift and diffusion of the CIR short-term rate in Eq. (32), i.e. $\mu = \mu_0$, $\sigma = \sigma_0$. We

assume, however, that in this auxiliary market, the final payoff is identically zero,

$$b_0 \equiv 0.$$

The price of the contract in the auxiliary market is, naturally, zero, $w_0(x, t) = 0$, and we also have $d(x) = 1$, $\delta(x, t) = 0$. By simple computations, then, we obtain that the approximation in Eq. (29) of Definition 1 collapses to:

$$w_N(x, t) = \sum_{n=0}^N \frac{(T-t)^n}{n!} d_n(x, t), \quad (34)$$

where:

$$\begin{aligned} d_0(x, t) &= 1, \quad d_1(x, t) = -x, \quad d_2(x, t) = -(\mu(x, t) - x^2), \\ d_3(x, t) &= -\frac{\partial \mu(x, t)}{\partial t} + \mu(x, t) \left(2x - \frac{\partial \mu(x, t)}{\partial x} \right) + \frac{1}{2} \sigma^2(x, t) \left(2 - \frac{\partial^2 \mu(x, t)}{\partial x^2} \right) + x(\mu(x, t) - x^2). \end{aligned}$$

Eq. (34) is a slight generalization to the power series expansion appearing in Chapman, Long and Pearson (1999, Proposition 3), and Wilmott (2003, p. 572). We assess the accuracy of this expansion to approximate the bond prices predicted by the CIR model, using the following parameter values: $\alpha = \alpha_0 = 0.06$, $\beta = \beta_0 = 0.10$ and $\sigma = 0.12$, and fixing the initial short-term rate level at $x = 10\%$. The figures for α , β and σ roughly match the average, standard deviation and first-order autocorrelation of the US overnight rate, using post-war data. Figure 3 plots the percentage pricing error arising for $N = 2, 4, 6, 8$, and 10. A truncation of Eq. (34) based on a few terms provides a quite accurate approximation to *short maturity* bond prices. Instead, many more terms are needed for the resulting approximation to work at longer maturities, as also documented by Kimmel (2008). As an example, the approximation based on only the first three terms works quite poorly for $T - t \geq 3$. We now turn to an expansion based on a richer auxiliary market, i.e. one for which the final payoff is not zero.

4.2.2 A Better Expansion: The Vasicek Model as Auxiliary Pricing Device

The results pertaining to the previous example can be improved, once we use a more informative auxiliary market, where the payoff of the bond is one, i.e. $b_0(x) = 1$, such that $d(x) = 0$. Consider, then, an auxiliary market, where the short-term rate is as in Vasicek (1977), and is solution to Eq. (33). The solution for the bond price, denoted with w_0 , is well-known as this is the simplest example of an exponential affine model. The mispricing function, δ , is now,

$$\delta(x, t; \theta_0) = (\mu(x, t) - \beta_0(\alpha_0 - x)) \frac{\partial w_0(x, t)}{\partial x} + \frac{1}{2} (\sigma^2(x, t) - \sigma_0^2) \frac{\partial^2 w_0(x, t)}{\partial x^2},$$

where now $\theta_0 = [\alpha_0 \ \beta_0 \ \sigma_0]^\top$ is the nuisance parameter vector arising from the use of the misspecified Vasicek model. Accordingly, the approximation to the CIR model is given, formally, by Eq. (20). As

in the option pricing problem of the previous section, we have a nuisance parameter vector to choose. In analogy with the Section 4.1, we could use: $\theta_N(x, t) = \arg \min_{\theta} (w_N(x, t; \theta) - w_0(x, t; \theta))^2$. Moreover, the two models, Vasicek and CIR, have a linear drift, which we perfectly match, by setting α_0 and β_0 equal to the numerical values we use for the CIR α and β , i.e. $\alpha = \alpha_0 = 0.06$ and $\beta = \beta_0 = 0.1$. The mispricing function δ , then, simplifies to:

$$\delta(x, t; \sigma_0) = \frac{1}{2} (\sigma^2 x - \sigma_0^2) \frac{\partial^2 w_0(x, t)}{\partial x^2}.$$

Figure 4 plots the approximation error against time-to-maturity for different values of N , and the same parameter values of the CIR model used for the numerical analysis summarized in Figure 3. We set the nuisance parameter σ_0 so as to equal $\sigma_0^2 = \sigma^2 x$, for a given level of the short-term rate, which corresponds to choosing $\theta_N(x, t) = \arg \min_{\theta} \delta^2(x, t; \theta)$ for all N . Finally, we set the current short-term rate to $x = 10\%$, as in the previous section. Compared to the approximation error of the simple expansion, the approximation based on the Vasicek model works considerably better, as it only needs a few terms to achieve a quite high level of precision. Numerical results not reported here confirm that the approximation works equally well for other initial values of the short-term rate, x .

4.3 Option Pricing with Stochastic Volatility

Finally, we study the numerical performance of our methods, by gauging the ability of the BS model, where stock volatility is constant, to approximate the solution to an European option price predicted by a model, where volatility is stochastic. Note, even if the BS model has constant volatility, our method, which in this section we choose to be relying on the BS model, does allow to feed information about stochastic volatility. The reason is that our power series expansions hinge upon the initial mispricing arising from the use of the BS model, and this mispricing is a function of the initial state, price and volatility. The expansions, then, deliver refinements that are increasingly more informative about stochastic volatility, as we shall illustrate.

Naturally, to approximate the unknown price in the market of interest, we might have wished to rely on an auxiliary market where stock volatility is random, rather than constant, as in the BS case. Our experiment to approximate a market with a given state space (that with stochastic volatility) through a market with a lower state space (that with constant volatility) serves the purpose to make a strong case for our methods. All in all, the numerical issue we wish to investigate here links to how many terms in the expansion are needed, in practice, to feed information about stochastic volatility so as to make our approximation reasonable.

Our benchmark model is that of Heston (1993), where, under the risk-neutral probability, the stock price, $\{S(t)\}$, is solution to,

$$\begin{cases} \frac{dS(t)}{S(t)} = r dt + \sqrt{v(t)} dW_1(t) \\ dv(t) = \kappa(\alpha - v) dt + \sigma \sqrt{v(t)} \left(\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right) \end{cases} \quad (35)$$

for four constant κ , α , σ and ρ , and two standard Brownian motions, W_1 and W_2 . In Eq. (35), $\{v(t)\}$ is the volatility process.

Let $w(S, v, t)$ be the price of the European option as of time $t \in [0, T]$, when the stock price is S and volatility is v . Come time T , $w(S, v, T) = \max\{S - K, 0\}$, where K is the strike price, as usual. Subject to this boundary condition, the pricing function, then, satisfies, $Lw(x, v, t) - rw(x, v, t) = 0$, where L is the infinitesimal generator operator associated to Eqs. (35):

$$Lw = \frac{\partial w}{\partial t} + rx \frac{\partial w}{\partial x} + \frac{1}{2}vx^2 \frac{\partial^2 w}{\partial x^2} + \kappa(\alpha - v) \frac{\partial w}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 w}{\partial v^2} + \sigma\rho vx \frac{\partial^2 w}{\partial x \partial v}. \quad (36)$$

As explained, we expand the price $w(x, v, t)$ in the stochastic volatility market around the auxiliary BS model, using the expansion set forth in Section 3 (Theorem 1 and Definition 1). Note that for our auxiliary model, the stock price is solution to:

$$\begin{cases} \frac{dS(t)}{S(t)} &= rdt + \sqrt{v(t)}dW_1(t) \\ dv(t) &= 0 \times dt + 0 \times (\rho dW_1(t) + \sqrt{1 - \rho^2}dW_2(t)) \end{cases}$$

where the initial condition for the volatility process is $v(0) = \sigma_0$, and σ_0 is constant.

While this way of re-writing the BS model appears more complicated than needed, it actually further illustrates our approach. Let us denote the BS pricing function as we did in Section 2.2, with $w^{\text{bs}}(x, t; \sigma_0)$, which is then the option price in our auxiliary market. The price difference, $\Delta w(x, v, t; \sigma_0) \equiv w(x, v, t) - w^{\text{bs}}(x, t; \sigma_0)$, satisfies,

$$L\Delta w(x, v, t; \sigma_0) - r\Delta w(x, v, t; \sigma_0) = \delta(x, v, t; \sigma_0), \quad \delta(x, v, t; \sigma_0) = \frac{1}{2}(v - \sigma_0^2)x^2 \frac{\partial^2 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^2}.$$

where $\Delta w(x, v, t; \sigma_0) = 0$, for all x and v .

Appealing to Theorem 1 and Definition 1, our N -order approximation to the true price $w(x, v, t)$, Heston's, is:

$$w_N(x, v, t; \sigma_0) = w^{\text{bs}}(x, t; \sigma_0) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x, v, t; \sigma_0), \quad (37)$$

where δ_n satisfies $\delta_{n+1}(x, v, t; \sigma_0) = L\delta_n(x, v, t; \sigma_0) - r\delta_n(x, v, t; \sigma_0)$, with $\delta_0 \equiv \delta$.

The approximation in Eq. (37) is seemingly identical to that in the extended BS-model of Section 2.2 (see Eq. (15)). In particular, the mispricing function, δ , has the same functional form as that in Eq. (12), and still bears the interpretation of an instantaneous hedging cost arising from the use of a wrong model, the BS model. However, the infinitesimal operator L in Eq. (36), which $\delta_n(x, t; \sigma_0)$ iterates upon, provides increasingly precise information about random volatility, as the iterations develop. To illustrate, consider the first-order approximation to the Heston's price, $w(x, v, t)$, viz

$$w_1(x, v, t; \sigma_0) = w^{\text{bs}}(x, t; \sigma_0) + \delta(x, v, t; \sigma_0)(T-t) + \frac{1}{2}\delta_1(x, v, t; \sigma_0)(T-t)^2. \quad (38)$$

The first term on the right-hand side is the BS price. The second, is the first adjustment, which is proportional to time-to-maturity, $T - t$, with proportionality factor equal to the mispricing function, δ . Intuitively, consider Eq. (28) in Theorem 1. The payoffs in both Heston's and BS markets are obviously the same and, hence, $d = 0$. Therefore, in the context of this section, it is only the third term on the right hand side of Eq. (28) that matters. By approximating the integrand of this third term with its value taken at t , we obtain the second term in Eq. (38). This approximation is quite rough: for example, the coefficients of the stochastic volatility process, κ , α , σ and ρ , do not enter δ . These coefficients enter the third, and final, term on the right hand side of Eq. (38). This term is product of a quadratic adjustment for time-to-maturity and the function δ_1 , obtained through one iteration upon the mispricing function δ . It equals:

$$\begin{aligned} \delta_1(x, v, t; \sigma_0) &= (v - \sigma_0^2) x^2 \varphi(x, v, t; \sigma_0) + \frac{1}{2} \kappa (\alpha - v) x^2 \frac{\partial^2 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^2} \\ &\quad + \sigma \rho v x \left(x \frac{\partial^2 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^2} + \frac{1}{2} x^2 \frac{\partial^3 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^3} \right) - r \delta(x, v, t; \sigma_0), \end{aligned}$$

where the function φ is defined as:

$$\begin{aligned} \varphi(x, v, t; \sigma_0) &\equiv \frac{1}{2} r x \frac{\partial^3 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^3} + \frac{1}{2} \frac{\partial^3 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^2 \partial t} + r \frac{\partial^2 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^2} \\ &\quad + v \left[\frac{1}{2} \frac{\partial^2 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^2} + x \left(\frac{\partial^3 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^3} + \frac{1}{4} x \frac{\partial^4 w^{\text{bs}}(x, t; \sigma_0)}{\partial x^4} \right) \right]. \end{aligned}$$

The first-order approximation in Eq. (38) is not expected to be accurate. It is only useful to illustrate, analytically, how the approximating price becomes more informative as we add new terms. For example, the volatility of volatility parameter, σ , enters δ_1 only through a correlation channel: if $\rho = 0$, σ does not enter $w_1(x, v, t; \sigma_0)$ anymore. Yet one iteration is sufficient to feed the approximating price with information about the parameters of the drift function of volatility, κ and α .

Figure 5 depicts the percentage approximation error, as a function of the current stock price, x , when: the current value of the volatility is such that $v = 0.05$, the strike price $K = 100$, time-to-maturity is one year, $T - t = 1$, the short-term rate $r = 10\%$, and the parameter values are roughly the same as those in Heston (1993): $\kappa = 2$, $\alpha = 0.04$, $\sigma = 0.10$, and $\rho = -0.5$. Finally, we set the nuisance parameter, the BS σ_0 , equal to \sqrt{v} . Figure 5 confirms that the first-order approximation, while improving over that obtained for $N = 0$ (i.e., that stemming from the use of the first two terms in Eq. (38)), still produces significant pricing errors. At the same time, the approximation in Eq. (37) considerably improves, and quite quickly, as we add new terms. With $N = 3$, for example, Eq. (38) provides a reasonable approximation to the Heston's price, with pricing errors amounting to less than one percent from the truth, over a realistic range of variation for the stock price. With $N = 4$, our approximation produces pricing errors as small as 0.2%, even for far-out-of-the money options.

5 Conclusion

We have developed a novel method to approximate the price of derivative assets in the context of multifactor continuous-time models. The idea underlying our approach is quite simple: given a model with no closed-form solution, select an “auxiliary” model, which has a closed-form solution, and expand the unknown price around the auxiliary one. We apply this method to a variety of asset pricing problems, spanning from the yield curve to stochastic volatility option pricing, and show that a truncation of this expansion up to a few terms is quite accurate. Naturally, our approach does not require any simulation, and once implemented, requires a trivial amount of computational time.

Our method can be used in a variety of related contexts, such as, for example, those pertaining to pricing exotic contracts through calibrated volatility surfaces. Calibrated volatility surfaces, even when smoothed, are unlikely to lead to closed-form expressions for the price of exotic derivatives, say a far-out-of-the money option. These prices, then, are typically solved through simulations. Our approach is a viable alternative. A second example where our approach has a potential is the estimation and calibration of asset pricing models. Estimation of continuous-time models typically centers around a set of conditional moments for asset returns, which can be readily obtained through simulation of our approximating pricing formulae.

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Figures

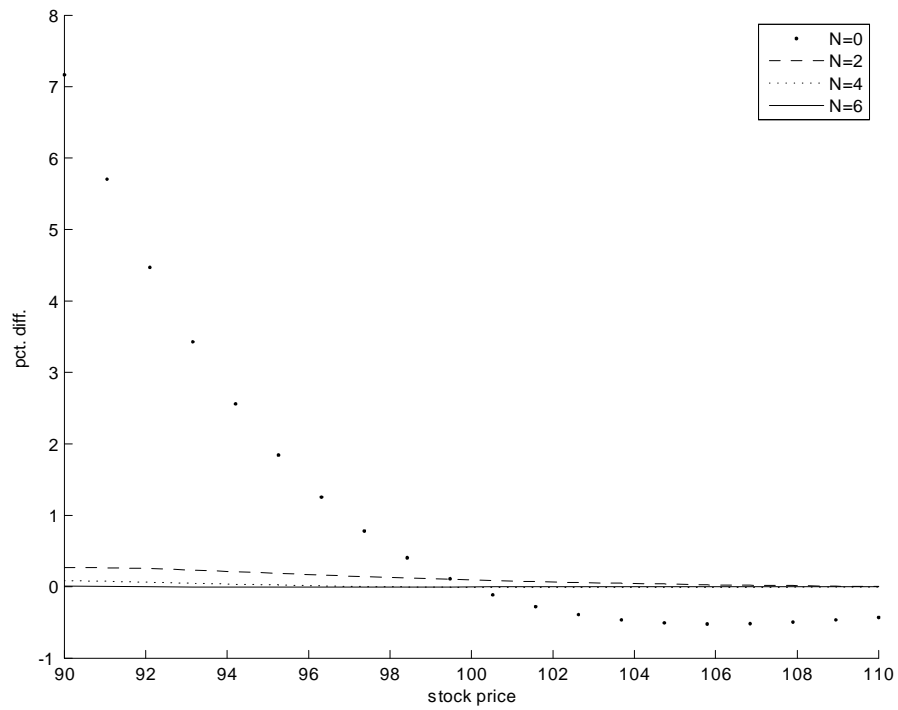


Figure 1: Percentage errors made by approximating the option price predicted by the CEV model through the Black-Scholes model, and N corrective terms. Strike price $K = 100$. Time-to-maturity: three months.

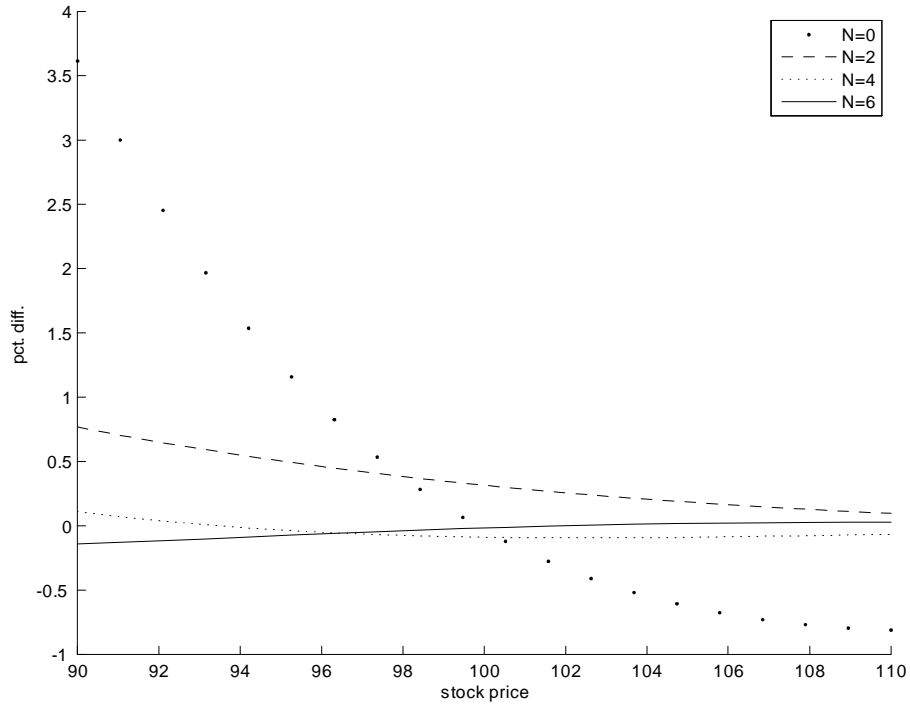


Figure 2: Percentage errors made by approximating the option price predicted by the CEV model through the Black-Scholes model, and N corrective terms. Strike price $K = 100$. Time-to-maturity: one year.

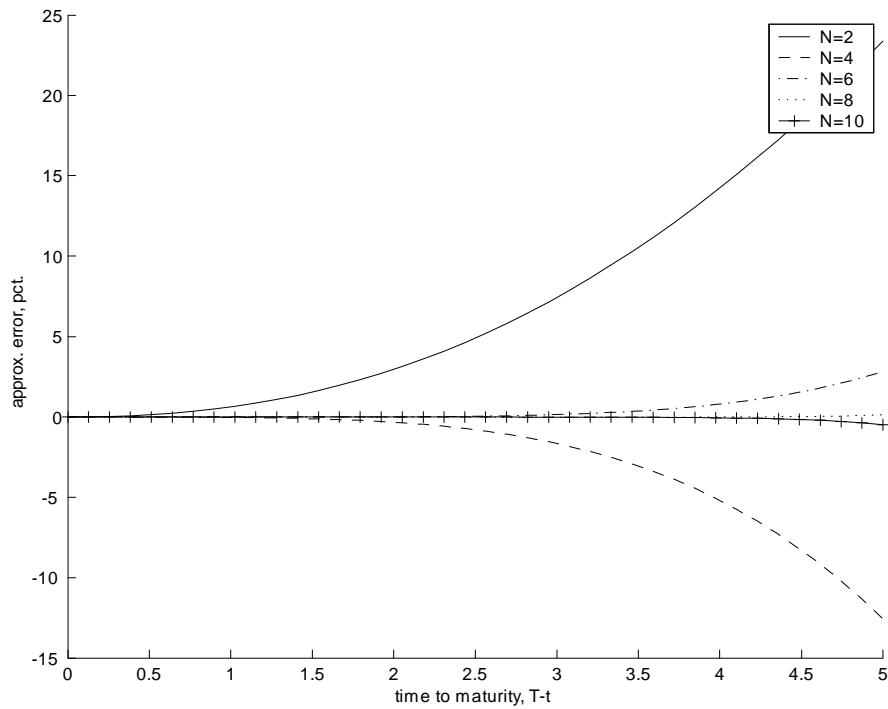


Figure 3: Percentage errors made by approximating the bond price predicted by the CIR model through a simple power expansion and N corrective terms. Current level of the short-term rate is 10%.

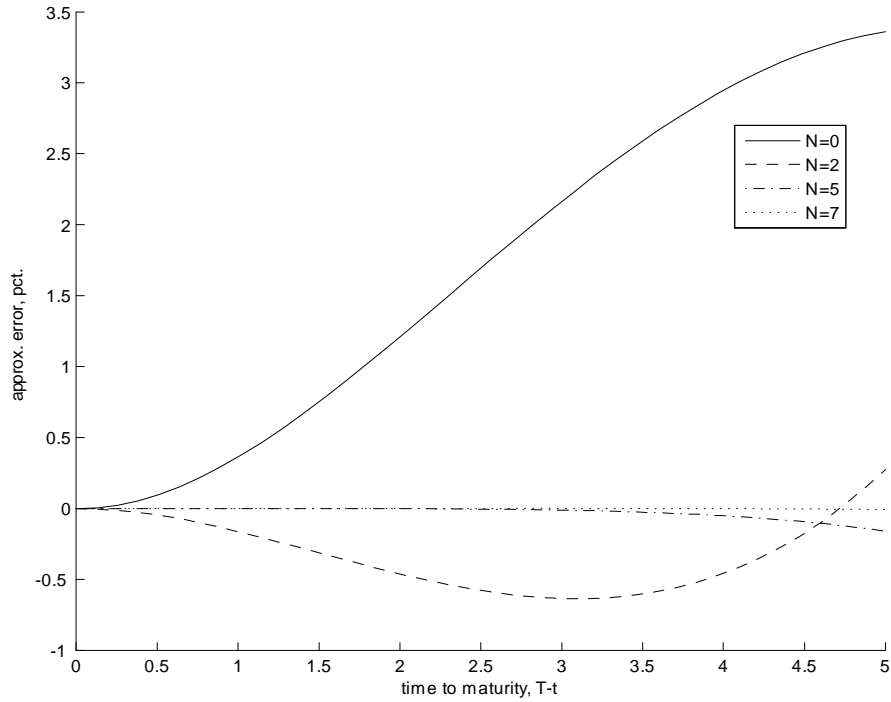


Figure 4: Percentage errors made by approximating the bond price predicted by the CIR model through the Vasicek model and N corrective terms. Current level of the short-term rate is 10%.

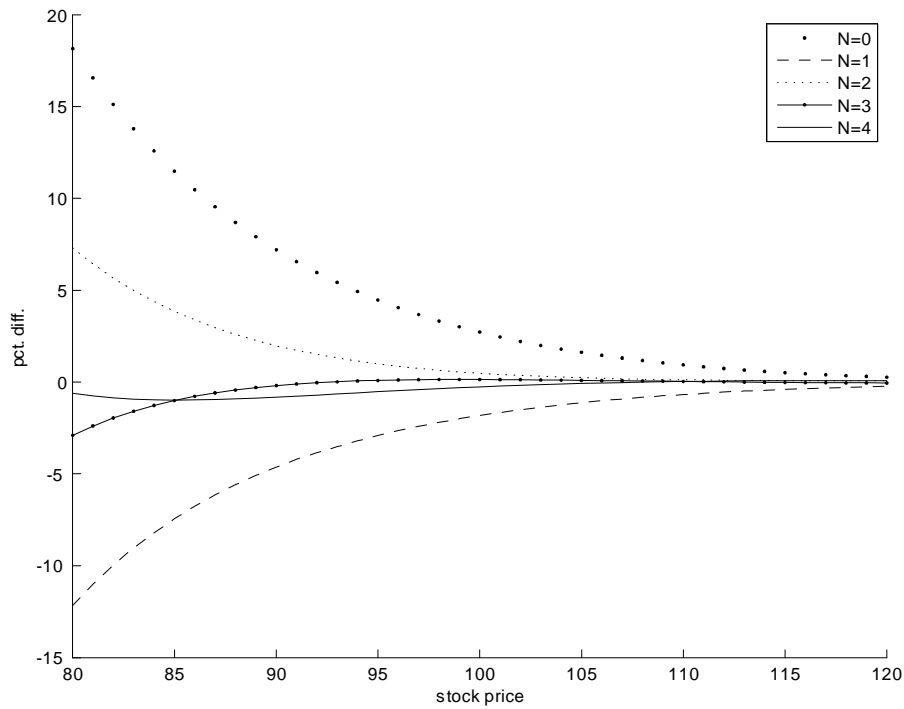


Figure 5: Percentage errors made by approximating the option price predicted by the Heston model through the Black-Scholes model, and N corrective terms. Strike price $K = 100$. Time-to-maturity: one year.

Technical Appendix

A The Expansion

This Appendix develops theoretical properties of the asset price approximation formula of Section 3, as given in Definition 1. Given the asset price $w(x, t)$ that solves Eq. (23), we provide conditions under which we can state error bounds for a fixed approximation, and also establish that $w_N(x, t) \rightarrow w(x, t)$ as $N \rightarrow \infty$, where $w_N(x, t)$ is our approximation of the asset price in Definition 1. The approach in this Appendix relies heavily on previous work that Schaumburg (2004) developed in a different context.

A.1 Properties

The next proposition establishes an error bound for the approximation, which holds for any fixed approximation order $N \geq 1$. We have:

Proposition A.1 *Assume that $\mu, \sigma^2 \in \mathcal{C}^{2N}(\mathbb{R}^d)$ and $d, \delta \in \mathcal{C}^{2(N+1)}(\mathbb{R}^d)$. Then w_N given in Definition 1 satisfies:*

$$|w(x, t) - w_N(x, t)| \leq E_N(x) \frac{(T-t)^{N+1}}{(N+1)!}, \text{ for all } (x, t) \in \mathbb{R}^d \times [0, T],$$

where

$$\begin{aligned} E_N(x) \equiv & \sup_{0 \leq s \leq T} \mathbb{E}_{x,t} [\|A^{N+1}d(x(s))\|] + \sup_{0 \leq s \leq T} \mathbb{E}_{x,t} [\|A^{N+1}\delta(x(s), s)\|] \\ & + \sup_{0 \leq s \leq T} \mathbb{E}_{x,t} [\|\partial^{N+1}\delta(x(s), s)/\partial s^{N+1}\|]. \end{aligned}$$

In particular, if μ, σ^2, d and δ are polynomially bounded, then:

$$E_N(x) \leq (1 + \|x\|^{q_N}) e^{c_N T},$$

for some constants c_N and q_N .

This result tells us that in great generality, the error decreases at a geometric rate uniformly over (x, t) in any compact interval as N increases. Florens-Zmirou (1989, Lemma 1) and Ait-Sahalia (2002) develop results similar to Proposition A.1 in different contexts.

Proposition A.1 is not informative about the asymptotic behavior of the error terms. In particular, we have not been able to establish bounds on q_N and c_N as N increases. To deal with the error terms for large N , we rely, instead, on results from the literature on operator theory. First, we introduce some additional notation and definitions. First, for a given operator A , we define its spectrum and resolvent as

$$\sigma(A) = \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not a bijection}\} \quad \text{and} \quad R_\lambda(A) = (\lambda - A)^{-1}, \quad \lambda \in \sigma(A).$$

Second, we introduce a function space \mathcal{H} of functions with domain \mathbb{R}^n and image in \mathbb{R} which is equipped with some function norm $\|\cdot\|_{\mathcal{H}}$. This can be chosen by the researcher given the specific asset pricing problem. We impose the following conditions on the spectrum and resolvent of the infinitesimal operator L of $\{x(t)\}$ in order to show that our power series expansion converges:

A.1 Given three constant $m, \omega > 0$ and $M \in (e^{-1}, \infty)$, the infinitesimal operator L given in Eq. (22) satisfies

$$\sigma(L) \subset \bar{\sigma} \equiv \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| > \pi/2 + m\},$$

and its resolvent satisfies $\|R_\lambda(L)\| \leq M/|\lambda|$ for $\lambda \in \mathbb{C} \setminus \bar{\sigma}(L)$.

A.2 There exists $\bar{\tau} > 0$ and $\phi_\delta, \phi_d \in \mathcal{H}$ such that $d : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ and $\delta : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}$ defined in Eqs. (26)-(27) satisfy:

$$\mathbb{E}[\phi_\delta(x(\bar{\tau})) | x(0) = x] = \delta(x, \bar{\tau}) \quad \text{and} \quad \mathbb{E}[\phi_d(x(\bar{\tau})) | x(0) = x] = d(x).$$

Also, the functions $\delta(x, t)$ and $R(x, t)$ are analytic, and $\sup_{x \in \mathbb{R}^n} |R(x, t)| < \infty$ for all $0 \leq t \leq T$.

Condition (A.1) relates to the infinitesimal operator and requires that its spectrum is within $\bar{\sigma}$. The second condition, (A.2), imposes conditions on the two functions d and δ , the conditional moments of which, we wish to expand about. It basically requires that each of the two functions can be matched through conditional moments. Both assumptions are abstract, and not easily verified for specific models. The following proposition develops more primitive conditions for (A.1) to hold that are typically met in many diffusion models.

Proposition A.2 *The generator L satisfies (A.1), under the following conditions:*

- (i) L has a transition density $p_t(y|x)$ with respect to Lebesgue measure.
- (ii) L has an invariant measure π satisfying

$$\pi(x) p_t(y|x) = \pi(y) p_t(x|y).$$

Conditions (i)-(ii) in Proposition A.2 are satisfied by many standard processes used in finance. Most diffusion models have a transition density, while the second condition is a generalization of time-reversibility. In particular, if the process is univariate and stationary, it is necessarily time-reversible and therefore satisfies the second condition. In conclusion, condition (A.1) holds under fairly weak conditions.

We are unaware of more primitive conditions for (A.2) to hold. For one example where (A.2) is not satisfied, we refer to Schaumburg (2004, Example 1), who provides additional discussion about this condition.

The theoretical foundations to the approximation in Definition 1 are in the following proposition:

Proposition A.3 *Let $\{x(t)\}_{t \geq 0}$ be a homogeneous diffusion process with infinitesimal operator L . Assume that L satisfies (A.1), and d , δ and R satisfy (A.2). Then for any $|t - T| < \bar{\tau}/(Me)$, where M and $\bar{\tau}$ are given in (A.1)-(A.2), we have:*

- (i) *The following equality holds:*

$$\mathbb{E} \left[e^{-\int_t^u R(x(s),s)ds} \delta(x(u),u) \middle| x(t) = x \right] = \sum_{n=0}^{\infty} \frac{(u-t)^n}{n!} \delta_n(x,t),$$

and

$$\int_0^T \mathbb{E} \left[e^{-\int_t^u R(x(s),s)ds} \delta(x(u),u) \middle| x(t) = x \right] du = \sum_{n=0}^{\infty} \frac{(T-t)^{n+1}}{(n+1)!} \delta_n(x,t),$$

where $\delta_0 \equiv \delta$ and

$$\delta_{n+1}(x,t) = L\delta_n(x,t) - R(x,t)\delta_n(x,t), \quad n \geq 0,$$

and similarly for the function d .

- (ii) *The approximation w_N given in Definition 1 satisfies, for all $|t - T| < \bar{\tau}/(Me)$,*

$$\|w_N(\cdot, t) - w(\cdot, t)\|_{\mathcal{H}} \rightarrow 0, \quad N \rightarrow \infty.$$

To establish Propositions A.1-A.3, we shall need to deal with the following Cauchy problem:

$$-\frac{\partial w(x,t)}{\partial t} = Aw(x,t) + b(x,t), \tag{A1}$$

for $(x,t) \in \mathbb{R}^d \times [0, T]$, where A is a general linear operator, and w satisfies the boundary condition:

$$w(x, T) = c(x).$$

We define the semigroup associated with A (see, e.g., Pazy, 1983) as:

$$U(t) = e^{tA},$$

and let $\mathcal{D}(A)$ denote the domain of A defined as the set of functions for which

$$A\phi(x,0) = \lim_{t \rightarrow 0} \frac{U(t)\phi(x,t) - \phi(x,0)}{t}$$

is well-defined. We note that Eq. (25) with time-homogenous coefficients can be cast in the same format as Eq. (A1), with

$$\begin{aligned} A\phi(x, t) &= L\phi(x, t) - R(x, t)\phi(x, t), \\ b(x, t) &= \delta(x, t), \quad c(x) = d(x). \end{aligned}$$

With this specification of A , we have:

$$U(t)\phi(x, t) = \mathbb{E} \left[\exp \left(- \int_0^t R(x(s), s) ds \right) \phi(x(t), t) \Big| x(0) = x \right].$$

It is easily seen that the solution to the inhomogenous problem of Eq. (A1) can be represented as:

$$w(x, t) = U(T-t)c(x) + \int_0^{T-t} U(s)b(x, s) ds. \quad (\text{A2})$$

Next, we obtain an approximate solution, w_N , through a series expansion of $U(t)$. In particular, we wish to give conditions under which $U(t)$ satisfies:

$$U(t)\phi(x) = e^{tA}\phi(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \phi(x), \quad (\text{A3})$$

in which case we define the approximation:

$$U_N(t)\phi(x) = \sum_{n=0}^N \frac{t^n}{n!} A^n \phi(x). \quad (\text{A4})$$

Suppose that the function $t \mapsto \phi(x, t)$ is analytic for all x , such that

$$\phi(x, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} B^k \phi(x, 0), \quad B\phi(x, t) \equiv \frac{\partial \phi(x, t)}{\partial t}.$$

Then,

$$U(t)\phi(x, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \phi(x, t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^{n+k}}{n!k!} A^n B^k \phi(x, 0) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (A+B)^n \phi(x, 0).$$

Thus, we shall use the following approximation:

$$U_N(t)\phi(x, t) = \sum_{n=0}^N \frac{t^n}{n!} (A+B)^n \phi(x, 0). \quad (\text{A5})$$

By plugging the two approximations in Eqs. (A4) and (A5) into Eq. (A2), we obtain:

$$w_N(x, t) = \sum_{n=0}^N \frac{(T-t)^n}{n!} A^n c(x) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} (A+B)^n b(x, 0).$$

The following proposition provides an upper bound to the approximation error for any given $N \geq 0$:

Proposition A.4 *Assume that the two functions $c(x)$ and $b(x, t)$ both belong to $\mathcal{D}(A^{N+1})$ and $t \mapsto b(x, t)$ is $N+1$ times differentiable. Then the approximation error satisfies:*

$$|w(x, t) - w_N(x, t)| \leq E_N(x) \frac{(T-t)^{N+1}}{(N+1)!}, \text{ for all } (x, t) \in \mathbb{R}^d \times [0, T],$$

where

$$E_N(x) = \sup_{0 \leq s \leq T} |A^{N+1}U(s)b(x, s)| + \sup_{0 \leq s \leq T} |A^{N+1}U(s)c(x)| + \sup_{0 \leq s \leq T} |B^{N+1}U(s)b(x, s)|.$$

Next, we establish conditions under which the error bound established in Proposition A.4 vanishes as $N \rightarrow \infty$. Intuitively, this result will go through if the power expansion in Eq. (A3) is valid. If the operator A was bounded, $\|A\| < \infty$, then the expansion would trivially hold. However, the infinitesimal operator is unbounded and, instead, we have to impose additional restrictions to verify the validity of the expansion. We impose restrictions in terms of the operator's spectrum and resolvent so as to ensure that A is a so-called analytic operator. In turn, these restrictions imply that the power expansion is valid. We have:

Proposition A.5 *Assume:*

(i) For some $\delta, \omega > 0$ and $M \in (e^{-1}, \infty)$:

$$\sigma(A) \subset \bar{\sigma}(A) \equiv \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| > \pi/2 + \delta\},$$

and $\|R_\lambda\| \leq M/|\lambda|$ for $\lambda \in \mathbb{C} \setminus \bar{\sigma}(A)$.

(ii) The functions $b(\cdot, \bar{\tau})$ and $c(\cdot)$ both lie in $U(\bar{\tau})\mathcal{H}$ for some $\bar{\tau} > 0$, i.e. there exists $\phi_b, \phi_c \in \mathcal{H}$ such that

$$U(\bar{\tau})\phi_b(x) = b(x, \bar{\tau}) \quad \text{and} \quad U(\bar{\tau})\phi_c(x) = c(x).$$

Moreover, $t \mapsto b(x, t)$ is analytic for all x .

Then for all $|t - T| < \bar{\tau}/(Me)$, where M and $\bar{\tau}$ are given in (i) and (ii):

$$\|w_N(\cdot, t) - w(\cdot, t)\|_{\mathcal{H}} \rightarrow 0, \quad N \rightarrow \infty.$$

Finally, note that the previous results only relate to time-homogenous diffusions. It would be of interest to derive results that also hold for time-inhomogenous diffusions, where drift and diffusion functions vary over time t . Heuristically, this task is tantamount to analyzing systems such as,

$$-\frac{\partial w(x, t)}{\partial t} = A(t)w(x, t) + b(x, t),$$

where the linear operator $A(t)$ is, now, time-inhomogenous. Unfortunately, there are still very few foundational results on the analyticity of this class of operators (for a few preliminary results, see Chapter 5 in Pazy, 1983). We have been unable to study how the previous propositions hold within such a more general setting.

A.2 Proofs

The regularity conditions underlying Theorem 1 are:

A.3 The two solutions, $w(x, t)$ and $w_0(x, t)$, exist and belong to $\mathcal{C}^{2,1}(\mathbb{R}^d \times [0, T])$. Furthermore, for some $C, q > 0$:

$$|w(x, t)| + |w_0(x, t)| \leq C(1 + \|x\|^q),$$

for all $x \in \mathbb{R}^d$ and $t \in [0, T]$.

A.4 The functions μ, μ_0, σ , and σ_0 are continuous and satisfy, for some $C > 0$:

$$\|\mu(x, t)\| + \|\mu_0(x, t)\| + \|\sigma(x, t)\| + \|\sigma_0(x, t)\| \leq C(1 + \|x\|),$$

for all $x \in \mathbb{R}^d$ and $t \in [0, T]$.

A.5 The function $R(x, t) \geq 0$ is continuous and satisfies the same growth condition as w and w_0 .

Proof of Theorem 1. Since w and w_0 are well-defined solutions to their partial differential equations, the difference, $\Delta w = w - w_0$, is a well-defined solution to the partial differential equation (25). We need to verify that the conditions for the Feynman-Kac formula are satisfied. We use the conditions of Karatzas and Shreve (1991, Theorem 5.7.6). First, given condition A.3, we have that $\Delta w(x, t)$ belongs to $\mathcal{C}^{2,1}(\mathbb{R}^d \times [0, T])$ and satisfies $|\Delta w(x, t)| \leq C(1 + \|x\|^q)$. Second, $\delta(x, t)$ and $R(x, t)$ are continuous and satisfy the same growth condition as Δw , due to conditions A.4 and A.5. Finally, the drift and diffusion terms μ and σ satisfy the necessary continuity and growth conditions. All the conditions in Karatzas and Shreve (1991, Theorem 5.7.6) hold. ■

Proof of Proposition A.1. This is a direct consequence of Proposition A.4 since under the conditions, d and δ clearly lies in the domain of L . Moreover,

$$A^{N+1}U(s)\delta(x,s) = \mathbb{E} [A^{N+1}\delta(x(s),s) | x(0) = x],$$

and similarly for the other term of $E_N(x)$ defined in Proposition A.4. This establishes the stated error bound. Finally, note that under the polynomial bounds, $\|A^{N+1}c(x(s),s)\| \leq C_N(\|x(s)\|^{q_N} + 1)$ for some $C_N > 0$ and $q_N \geq 1$, and we then apply Friedman (1975, Theorem 5.2.2-5.2.3) to obtain

$$\mathbb{E} [|A^{N+1}\delta(x(s),s)| | x(0) = x] \leq C_N (\mathbb{E} [\|x(s)\|^{q_N} | x(0) = z] + 1) \leq (1 + \|x\|^{q_N}) e^{c_N s},$$

for some constants $c_N, q_N > 0$. Similarly for the other term in $E_N(x)$. ■

Proof of Proposition A.2. It follows from Schaumburg (2004, Lemma 2.2) that L satisfies (A.1) under the two conditions stated in the proposition. ■

Proof of Proposition A.3. The result will follow from Proposition A.5 if we can verify that (A.1)-(A.2) imply (i)-(ii) of that Proposition. It is easily seen that, given the form of $U(t)$ for the choice of A , (A.2) implies (ii). To verify (i), we apply Pazy (1983, Theorem 3.2.1) which will yield the desired result if we can show that the domain of L is contained in that of the operator F , defined as $F\phi(x,t) = R(x,t)\phi(x)$, with $\mathcal{D}(L) \subset \mathcal{D}(F)$, and that,

$$\|F\phi(\cdot,t)\|_{\mathcal{H}} \leq c_1 \|L\phi(\cdot,t)\|_{\mathcal{H}} + c_2 \|\phi(\cdot,t)\|_{\mathcal{H}},$$

for some constants c_1 and c_2 . But clearly, $\mathcal{D}(F)$ contains all twice-differentiable functions and the above inequality follows by the fact that $\sup_{x \in \mathbb{R}^n} |R(x,t)| < \infty$. ■

Proof of Proposition A.4. By definition,

$$U(t)\phi(x,t) = \phi(x,t) + \int_0^t AU(s)\phi(x,s) ds.$$

Using this identity iteratively, we obtain

$$\begin{aligned} U(t)\phi(x,t) &= \phi(x,t) + \int_0^t AU(t_1)\phi(x,t) dt_1 \\ &= \phi(x,t) + \int_0^t A \left[\phi(x,t) + \int_0^{t_1} U(t_2)\phi(x,t_2) \right] dt_2 dt_1 \\ &= \phi(x,t) + tA\phi(x,t) + \int_0^t \int_0^{t_1} AU(t_2)\phi(x,t_2) dt_2 dt_1 \\ &\quad \vdots \\ &= \sum_{n=0}^N \frac{t^n}{n!} A^n \phi(x,t) + E_N(x,t), \end{aligned}$$

where

$$E_N(x,t) = \frac{1}{N!} \int_0^t \int_0^{t_1} \dots \int_0^{t_{N+1}} A^{N+1}U(t_{N+1})\phi(x,t_{N+1}) dt_{N+1} \dots dt_1.$$

The approximation error $E_N(x,t)$ is bounded by:

$$\begin{aligned} |E_N(x,t)| &= \frac{1}{N!} \left| \int_0^t \int_0^{t_1} \dots \int_0^{t_{N+1}} A^{N+1}U(t_{N+1})\phi(x,t_{N+1}) dt_{N+1} \dots dt_1 \right| \\ &\leq \frac{1}{N!} \int_0^t (t-s)^N |A^{N+1}U(s)\phi(x,s)| ds \\ &\leq \sup_s |A^{N+1}U(s)\phi(x,s)| \times \frac{1}{(N+1)!} t^{N+1}. \end{aligned}$$

Next, by an N -th order Taylor expansion of b , there exists $\bar{s} \in [0, s]$ such that

$$b(x, s) - \sum_{k=0}^N \frac{t^k}{k!} B^k b(x, 0) = \frac{t^{N+1}}{(N+1)!} B^{N+1} b(x, \bar{s}).$$

Using these results,

$$\begin{aligned} & |w(x, t) - w_N(x, t)| \\ & \leq |[U - U_N](T-t)c(x)| + \int_0^{T-t} |[U - U_N](s)b(x, s)| ds + \int_0^{T-t} |U(s)[b - b_N](x, s)| ds \\ & \leq \frac{(T-t)^{N+1}}{(N+1)!} \sup_s |A^{N+1}U(s)c(x)| + \frac{(T-t)^{N+2}}{(N+2)!} \sup_s |A^{N+1}U(s)b(x, s)| \\ & \quad + \frac{(T-t)^{N+2}}{(N+2)!} \sup_s |B^{N+1}U(s)b(x, s)|. \end{aligned}$$

■

Proof of Proposition A.5. We apply Pazy (1983, Theorem 2.5.2) to obtain that the range of $U(t)$ is dense in $\mathcal{D}(A^\infty)$ and, hence, in \mathcal{H} under (i). Proposition A.4 supplies an upper bound to the approximation, for a given N . By the same arguments as in Schaumburg (2004, Proof of Theorem 2.1), we obtain that

$$\left\| (T-t)^{N+1} A^{N+1} U(s) c \right\| \rightarrow 0, \quad \left\| (T-t)^{N+1} A^{N+1} U(s) b(\cdot, s) \right\| \rightarrow 0,$$

as $N \rightarrow \infty$ for all $(T-t) < \bar{\tau}/(Me)$. Moreover, we have that for an analytical function, $(T-t)^{N+1} B^{N+1} b(\cdot, s) \rightarrow 0$. By dominated convergence, then,

$$\left\| (T-t)^{N+1} U(s) B^{N+1} b(\cdot, s) \right\| \rightarrow 0.$$

Hence, the bound goes to zero as $N \rightarrow \infty$. ■

B Approximations to Second Order Derivatives of Asset Prices

For $k = 2$, the recursive scheme needed to compute the second-order partial derivatives of $d_n(x, t)$ and $\delta_n(x, t)$ with respect to x , is: $d_0^{(2)}(x, t) = \partial^2 d(x) / \partial x^2$, $\delta_0^{(2)}(x, t) = \partial^2 \delta(x, t) / \partial x^2$ and,

$$\begin{aligned} d_n^{(2)}(x, t) &= L d_{n-1}^{(2)}(x, t) - R(x, t) d_{n-1}^{(2)}(x, t) + 2L^{(1)} d_{n-1}^{(1)}(x, t) - 2 \frac{\partial R(x, t)}{\partial x} d_{n-1}^{(1)}(x, t) \\ &\quad + L^{(2)} d_{n-1}(x, t) - \frac{\partial^2 R(x, t)}{\partial x^2} d_{n-1}(x, t) \end{aligned}$$

$$\begin{aligned} \delta_n^{(2)}(x, t) &= L \delta_{n-1}^{(2)}(x, t) - R(x, t) \delta_{n-1}^{(2)}(x, t) + 2L^{(1)} \delta_{n-1}^{(1)}(x, t) - 2 \frac{\partial R(x, t)}{\partial x} \delta_{n-1}^{(1)}(x, t) \\ &\quad + L^{(2)} \delta_{n-1}(x, t) - \frac{\partial^2 R(x, t)}{\partial x^2} \delta_{n-1}(x, t), \end{aligned}$$

where

$$L^{(2)} \phi(x, t) = \sum_{i=1}^d \frac{\partial^2 \mu_i(x, t)}{\partial x^2} \frac{\partial \phi(x, t)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 \sigma_{ij}^2(x, t)}{\partial x^2} \frac{\partial^2 \phi(x, t)}{\partial x_i \partial x_j}.$$

C Equivalence between Moment and Density Expansions

C.1 Expansions of Risk-Neutral Probabilities

We prove the equality stated in Eq. (31). In the process, we also obtain a direct representation of the difference between the conditional densities of the true and the auxiliary model, the “transition discrepancy.” First, note that the two transition densities solve the backward Kolmogorov equation:

$$Lp(y, T|x, t) = 0, \quad L_0p_0(y, T|x, t) = 0,$$

with boundary conditions $p(y, T|x, T) = p_0(y, T|x, T) = \text{Dir}(y - x)$, where $\text{Dir}(\cdot)$ is the Dirac’s function. Using the same arguments as those in Section 3, it is easily seen that the transition discrepancy, Δp , is solution to:

$$L\Delta p(y, T|x, t) + \tilde{\delta}(y, T|x, t) = 0,$$

with boundary condition $\Delta p(y, T|x, T) = 0$, where the adjustment term $\tilde{\delta}(y, T|x, t)$ is given by:

$$\tilde{\delta}(y, T|x, t) = (L_0 - L)p_0(y, T|x, t) = \Delta\mu(x, t) \frac{\partial p_0(y, T|x, t)}{\partial x} + \frac{1}{2}\Delta\sigma^2(x, t) \frac{\partial^2 p_0(y, T|x, t)}{\partial x^2}.$$

By the Feynman-Kac representation theorem,

$$\Delta p(y, T|x, t) = \int_t^T \mathbb{E}_{x,t}[\tilde{\delta}(y, T|x(s), s)] ds. \quad (\text{C1})$$

Substituting the right hand side of Eq. (C1) back into Eq. (30),

$$\begin{aligned} w(x, t) &= w_0(x, t) + \int_{\mathbb{R}^d} b(y) \Delta p(y, T|x, t) dy \\ &= w_0(x, t) + \int_t^T \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(y) \tilde{\delta}(y, T|z, s) p(z, s|x, t) dy dz \right] ds. \end{aligned} \quad (\text{C2})$$

Finally, using that

$$\frac{\partial^k w_0(x, t)}{\partial x^k} = \int_{\mathbb{R}^d} b(y) \frac{\partial^k p_0(y, T|x, t)}{\partial x^k} dy, \quad k \geq 0,$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(y) \tilde{\delta}(y, T|z, s) p(z, s|x, t) dy dz &= \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} b(y) \frac{\partial p_0(y, T|z, s)}{\partial z} dy \right] \Delta\mu(z, s) p(z, s|x, t) dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} b(y) \frac{\partial^2 p_0(y, T|z, s)}{\partial z^2} dy \right] \Delta\sigma^2(z, s) p(z, s|x, t) dz \\ &= \int_{\mathbb{R}^d} \Delta\mu(z, s) \frac{\partial w_0(z, s)}{\partial x} p(z, s|x, t) dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \Delta\sigma^2(z, s) \frac{\partial w_0^2(z, s)}{\partial x^2} p(z, s|x, t) dz \\ &= \mathbb{E}_{x,t}[\delta(x(s), s)], \end{aligned}$$

where δ is as in Eq. (27).

Note that the above representation of the transition discrepancy, in terms of a conditional moment, gives rise to an alternative approximation scheme. Precisely, the right hand side of Eq. (C1) can be approximated by

$$\Delta p_N(y, T|x, t) = \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} L\tilde{\delta}(y, T|x, t).$$

Plugging the previous expression into the integral in Eq. (C2), we obtain

$$\tilde{w}_N(x, t) = w_0(x, t) + \sum_{n=0}^N \frac{(T-t)^{n+1}}{(n+1)!} \int_{\mathbb{R}^d} b(y) L^n \tilde{\delta}(y, T|x, t) dy.$$

However, as noted in the main text, this type of approximation involves the computation of N Riemann integrals, $\int_{\mathbb{R}^d} b(y) L^n \tilde{\delta}(y, T|x, t) dy$, $n = 1, \dots, N$.

C.2 Relation to Saddlepoint Approximations

We develop an interpretation of saddlepoint approximations as a special choice of the approximating auxiliary pricing models that we have been discussed in the main text. For simplicity, we maintain the assumption in the main text, that $R(x, t) = c(x, t) \equiv 0$. Consider the first-order saddlepoint approximation of the transition density $p(y, T|x, t)$. As shown by Aït-Sahalia and Yu (2006, Theorem 1), this first-order approximation is, for $d = 1$:

$$p_0^{(1)}(y, T|x, t; \theta_0) = \frac{1}{\sqrt{2\pi\sigma_0^2(T-t)}} \exp\left(\frac{(y-x-\mu_0(T-t))^2}{2\sigma_0^2(T-t)}\right), \quad (\text{C3})$$

where $\theta_0 = (\mu_0, \sigma_0^2)$, and is chosen so as to match the initial values of the drift and diffusion of the true model, i.e. $\theta_0 = \theta_0(x) = (\mu(x), \sigma^2(x))$. The transition density in Eq. (C3) is that of an arithmetic Brownian motion

$$dx_0(t) = \mu_0 dt + \sigma_0 dW(t). \quad (\text{C4})$$

In the context of our method, then, the arithmetic Brownian motion in Eq. (C4) corresponds to an auxiliary model about which we expand the true model. In other words, the auxiliary transition density is given by $p_0(y, T|x, t) = p_0^{(1)}(y, T|x, t; \theta_0(x))$. By Theorem 1, then, we can give an explicit expression for the error involved whilst using a first-order saddlepoint approximation instead of the true density. It is:

$$w(x, t) = w_0(x, t) + \int_t^T \mathbb{E}_{x,t}[\delta(x(s), s)] ds, \quad (\text{C5})$$

where δ as in Eq. (27), with $\mu_0(x) = \mu_0$ and $\sigma_0^2(x) = \sigma_0^2$.

Therefore, higher-order saddlepoint approximations can be interpreted as those taking into account the second term given in Eq. (C5). For example, the second-order saddlepoint approximation of p is:

$$p_0^{(2)}(y, T|x, t) = p_0^{(1)}(y, T|x, t) \frac{1 + c_1(y, T|x, t)(T-t)}{1 + d_{1/2}(y, T|x, t)\sqrt{T-t} + d_1(y, T|x, t)(T-t)},$$

where the expressions for c_1 , $d_{1/2}$ and d_1 are in Aït-Sahalia and Yu (2006, Theorem 2). In the context of our framework, this second-order approximation can be interpreted as an adjustment for the presence of the second term in Eq. (C5). Namely, the second-order approximation implies the following approximation to the difference between the true and the auxiliary risk-neutral conditional density,

$$\Delta \tilde{p}^{(2)}(y, T|x, t) \equiv p_0^{(1)}(y, T|x, t) \left[\frac{1 + c_1(y, T|x, t)(T-t)}{1 + d_{1/2}(y, T|x, t)\sqrt{T-t} + d_1(y, T|x, t)(T-t)} - 1 \right].$$

From this expression, we obtain the following second-order approximation for $w(x, t)$,

$$\tilde{w}_2(x, t) = w_0(x, t) + \int_{\mathbb{R}^d} b(y) \Delta \tilde{p}^{(2)}(y, T|x, t) dy.$$

As explained in the main text, the advantage of the approximation in Definition 1 over \tilde{w}_2 , is the availability of closed-form approximations for the error term that are easily implemented. Finally, the previous discussion relies on saddlepoint approximations, obtained through a Gaussian basis. If the true model is far from the Gaussian, lower-order approximations will be inaccurate, and a non-Gaussian basis is more appropriate, as suggested by Aït-Sahalia and Yu (2006, Section 3.3). Within our framework, a non-Gaussian basis translates into a non-Gaussian auxiliary model. Again, we can develop an explicit representation of, with closed-form approximation for, the mispricing arising while using this auxiliary model

to approximate asset prices.