A Simple Multi-Factor “Factor Adjustment” for the Treatment of Credit Capital Diversification

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Abstract

We present a simple adjustment to the single-factor credit capital model, which recognizes the diversification from a multi-factor credit setting. The model can be applied to extend the Basel II regulatory framework to a general multi-factor setting, thus allowing for more accurate modeling of diversification for portfolios across various asset classes, sectors and regions, and in particular within mixed portfolios in developed and emerging economies.

We introduce the concepts of a *diversification factor* at the portfolio level, as well as *marginal diversification factors* at the obligor or sub-portfolio level, which further capture diversification contributions to the portfolio. We estimate the diversification factor for a family of multi-factor models, and show that it can be expressed as a function of two parameters that broadly capture the size concentration and the average cross-sector correlation. This model supports an intuitive capital allocation methodology, where the diversification contribution of a given sector can be further attributed to three components: the overall portfolio diversification, the relative size of the sector to the overall portfolio, and its cross-sector correlation. The estimated diversification factor can be tabulated for the implementation of credit portfolio decision management support tools as well as potential regulatory applications. As a risk management tool, it can be used to understand concentration risk, capital allocation and sensitivities, as well as to compute "real-time" marginal risk contributions for new deals or portfolios.

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1. Introduction

Minimum credit capital requirements under the new Basel II Capital Accord (Basel Committee of Banking Supervision, 2003) are based on the estimation of the 99.9% systemic credit risk for a portfolio (the risk of an asymptotically fine-grained portfolio) under a one-factor Merton type credit model. This results in a closed form solution, which provides additive risk contributions for each position and that is also easy to implement. The two key limitations of this model are that it measures only systemic credit risk, and it might not recognize the full impact of diversification.

The first shortcoming has been addressed in an analytical manner, most notably with the introduction of a granularity adjustment (Gordy 2003, Wilde 2001, Martin and Wilde 2002). The second problem is perhaps more difficult to address analytically but has greater impact, especially for institutions with broad geographical and asset diversification. Diversification is one of the key tools for managing credit risk, and it is vital that the credit portfolio framework, used to calculate and allocate credit capital, effectively models portfolio diversification effects.

Portfolio granularity and full diversification within a multi-factor setting can be effectively addressed within a simulation-based credit portfolio framework. However, there are benefits for seeking analytical, closed-form, models both for regulatory applications as well as for credit portfolio management. While the use of credit portfolio simulation-based models is now widespread, they are computationally intensive and may not provide further insights into sources of risk. They are also not efficient for the calculation of various sensitivities, or provide practical solutions for real-time decision support. Furthermore, the accurate calculation of marginal capital contributions in a simulation framework has proven to be a difficult computational problem, which is currently receiving substantial attention from both academics and practitioners (see Kalkbrener et al. 2004, Merino and Nyfeler, 2004, Glasserman 2005). Analytical or semi-analytical methods generally provide tractable solutions for capital contributions (c.f. Martin et al. 2001, Kurth and Tasche 2003).

In terms of multi-factor credit portfolio modeling, Pykhtin (2004) recently obtains an elegant, analytical multi-factor adjustment, which extends the granularity adjustment technique of Gordy, Martin and Wilde. This method can also be used quite effectively to compute capital contributions numerically (given its closed form solution to compute portfolio capital). However, the closed-form expressions for capital contributions can be quite intricate.

In this paper, we present an adjustment to the single-factor credit capital model, which recognizes the diversification from a multi-factor setting and which can be tabulated easily for risk management decision support and potential regulatory application. The objective is to obtain a simple and intuitive approximation, based only on a small number of parameters, and which is perhaps less general and requires some calibration work.

To develop the model, we introduce the concept of a *diversification factor*, *DF*, defined as

$$DF = \frac{EC^{mf}}{EC^{Sf}} \quad , \quad DF \leq 1 \tag{1}$$

where EC^{mf} denotes the diversified economic capital from a multi-factor credit model and EC^{sf} is the economic capital arising from the single-factor model.

For a given α percentile level (e.g. α = *0.1%*), we seek an approximation to the multi-factor economic capital of the form

$$EC^{mf}(\alpha; \cdot\,) \approx DF(\alpha; \cdot\,)\, EC^{sf}(\alpha) \tag{2}$$

with $DF(\alpha; \cdot\,) \leq 1$ a scalar function of a small number of (yet to be determined) parameters. A simple expression of the form (2) basically allows us to express the diversified capital as a function of the "additive" bottoms-up capital from a one-factor model (e.g. the Basel II model), and to tabulate the diversification factor (as a function of say two or three parameters). For potential regulatory use, we may also seek a *conservative* parameterization of equation (2).

We estimate the diversification factor for a family of multi-factor models, and show that it can be expressed as a function of two parameters that broadly capture the *size concentration* and the *average cross-sector correlation*.

The diversification factor provides a practical risk management tool to understand concentration risk, capital allocation and correlations, and various capital sensitivities. For this purpose, we further introduce *marginal diversification factors* at the obligor or sub-portfolio level, which further account for the diversification contributions to the portfolio.[5] The model (2) supports an intuitive capital allocation methodology, where the diversification contribution of a given sector can be further attributed to three components: the overall portfolio diversification, the sector's relative size to the overall portfolio, and its cross-sector correlation. Finally, for a given portfolio, we can readily fit the model to a full multi-factor internal credit portfolio model (which may be simulation based). The resulting implied parameters of the model provide simple risk and sensitivity indicators, which allow us to understand the sources of risk and concentration in the portfolio. The fitted model can then be used as a practical tool for real-time computation of marginal capital for new loans or other credit instruments, and for further sensitivity analysis.

The rest of the paper is organized as follows. We first motivate the use of multi-factor models through an empirical analysis of possible ranges of asset correlations across various economies, and particularly across developed and emerging countries. We then introduce the underlying credit model, the diversification factor and its general analytical justification, and the resulting capital allocation methodology. Thereafter, we show how the diversification factor can be estimated numerically using a full credit portfolio model and Monte Carlo simulations. We provide several parameterization exercises in the context of the Basel II formulae for wholesale exposures. Finally, we discuss the application of the model as a risk management tool, in conjunction with an internal multi-factor economic capital model, to understand concentration risk and capital allocation, as well as for real-time marginal economic capital calculation.

2. Motivation – Example: Estimating Correlations in Developed and Emerging Economies

Diversification is one of the key tools for managing credit risk and optimally allocating credit capital. The accurate modeling of diversification has important consequences for institutions with broad geographical and asset coverage, as well as for those actively managing credit risk. This is especially true within international banks, with substantial credit activities across different

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[5] This paper is closely related to Tasche (2006) who further presents a mathematical foundation for the diversification factor and diversification contributions. The author presents a two-dimensional example which has an analytical solution, and more generally the contribution expressions require integrals of dimension *N-1*, for problems of dimension *N*.

countries. Thus, many institutions today have in production either internally developed or commercial multi-factor credit portfolio models across their wholesale and retail portfolios.

In this section, we motivate the importance of using multi-factor models through an empirical correlation analysis. As is common practice, we use equity correlations as a proxy for asset correlations (see for example Gupton et al 1997**)**. Although there are many known limitations for using equity correlations, our objective is only to provide an intuitive picture for the ranges of asset correlations, as well as for the number of factors required to model these within and across developed and emerging economies**.** Thus, the broad, qualitative, conclusions we draw from the analysis should not be impacted by this crude approximation.

We use as proxies the stock market indices of the different countries. Table 1 displays the average correlations between countries within developed and emerging economies and across both groups on the basis of monthly returns over a period of 7 years (1996-2003). The average correlation between the indices of developed economies stands at around 74%, whereas the average correlation between developed and emerging economies, as well as between emerging economies, is around 40%. The Appendix further presents the detailed correlation matrix.

	Developed economies	Emerging economies
Developed economies	0.74	0.41
Emerging economies	0.41	0.40

Table 1. Average asset correlations from stock market indices

Alternatively, we can use aggregate indices instead of using individual market indices for each country[6]. In this case, the correlation between the two aggregated global indices is 61%, which is still not very high in spite of the fact that considering general indices tends to raise correlations.

To give a better characterization of the multi-factor nature of the problem, we perform a principal components analysis (PCA) of the individual stock market index returns. Table 2 presents the percentage of variance explained by the factors resulting from the PCA. A single factor accounts

[6] Based on series of monthly returns over 7 years of the S&P Emerging Market and Morgan Stanley Developed Markets Indices (1996-2003).

for 77.5% of the variability of the developed markets, and three factors are required to explain more than 90%. In contrast, the first factor only explains about 47% of the variability of emerging market indices and seven factors are required to explain more than 90%. Although the single-factor model is not a satisfactory simplification in either of the two cases, this model is even further removed from reality in the case of emerging economies.

	%VARIABILITY		%ACCUMULATED VARIABILITY	
	DEVELOPED	**EMERGING**	**DEVELOPED**	**EMERGING**
Factor 1	77.5	46.7	77.5	46.7
Factor 2	8.3	14.2	85.8	60.9
Factor 3	5.4	10.7	91.2	71.7
Factor 4	3.1	7.2	94.3	78.8
Factor 5	2.2	5.9	96.6	84.7
Factor 6	1.5	4.6	98.1	89.2
Factor 7	1.1	4.3	99.2	93.5
Factor 8	0.8	3.3	100.0	96.9
Factor 9		3.1		100.0

Table 2. PCA analysis of stock market indices

To complement the previous analysis, we estimate the correlation between the PCA factors for developed and emerging economies. Table 3 shows the correlation structure of the first three principal components for each group (with F_i and G_i denoting the factors for developed countries and emerging countries, respectively).

Correlations between factors

	G_1	G_2	G_3
F_1	0.41	-0.05	0.70
F_2	0.32	0.44	0.45
F_3	-0.03	-0.81	0.39

Table 3. Correlation between PCA factors

In summary, there are multiple factors that affect developed and emerging economies and, moreover, these factors are not the same in both cases. It is thus important to consider a multi-factor model for dealing suitably with financial entities that have investments in both developed and emerging economies.

Simple Two-Dimensional Diversification Example

Consider the case of a corporate portfolio consisting of one sub-portfolio with exposures in a developed economy, with stronger credit standing, and a second one in an emerging economy, with weaker average credits. As an example, Table 4 shows the calculation of the economic capital required by a portfolio with 94% of exposures in the developed economy (portfolio with *PD* of 2.5%), and the remaining 6% in the emerging economy (average *PD* of 5.25%). We assume an average *LGD* of 50% . The total capital required (excluding expected loss) is 9.37%, using the Basel II model (single-factor). Under a two-factor model with a correlation of 60%, the capital requirements fall to 9.01%. This is a reduction of about 4% of capital due to diversification or, alternatively, a factor adjustment of 0.96 (i.e. 9.01% = 9.37% x 0.96).

	Portfolio 1	*Portfolio 2*	*Total*
Average prob. of default	2.5%	5.25%	2.7%
Percentage of exposure	94%	6%	100%
Loss given default	50%	50%	50%
Average correlation	15%	13%	
Expected loss	0.0119	0.0015	1.34%
Capital (without EL)	0.0871	0.0066	9.37%
Total	0.0991	0.0081	10.71%

Total Entity	**One factor model**	**Two factor model**	**Reduction factor**
Expected loss	1.34%	1.34%	
Capital (without EL)	9.37%	9.01%	96%
Total	10.71%	10.35%	

Table 4. Example: two-factor credit portfolio

3. A Model for the Diversification Factor

We first introduce the underlying credit model. We then define the concepts of the diversification factor and the capital diversification index, and outline the estimation method. Finally we discuss capital allocation and risk contributions within the model.

Underlying Credit Model and Stand-Alone Capital

Consider a single-step model with K sectors (each of these sectors can represent an asset class or geography, etc.). For each obligor j in a given sector k, the credit losses at the end of the horizon (say, one year) are driven by a single-factor Merton model [7]. Obligor *j* defaults when a continuous random variable Y_j, which describes its creditworthiness, falls bellow a given threshold at the given horizon. If we denote by PD_j the obligor's (unconditional) default probability and assume that the creditworthiness is standard normal, we can express the default threshold by $N^{-1}(PD_j)$.

The creditworthiness of obligor *j* is driven by a single systemic factor:

$$Y_j = \sqrt{\rho_k}\, Z_k + \sqrt{1-\rho_k}\, \varepsilon_j \tag{3}$$

where Z_k is a standard Normal variable representing the systemic factor for sector *k,* and the ε_j are independent standard Normal variables representing the idiosyncratic movement of an obligor's creditworthiness. While in the Basel II model all sectors are driven by the same systemic factor *Z,* here each sector can be driven by a different factor.

We assume further that the systemic factors are correlated through a single macro-factor, *Z*

$$Z_k = \sqrt{\beta}\, Z + \sqrt{1-\beta}\, \eta_k \quad , \qquad k = 1,\ldots,K \tag{4}$$

where η_k are independent standard Normals. For simplicity we have assumed a single correlation parameter for all the factors (as we seek a simple parametric solution). Later, we allow for this parameter β to be more generally an average factor correlation for all the sectors.

For ease of notation, assume that for obligor *j* has a single loan with loss given default and exposure at default given by LGD_j, EAD_j respectively. As shown in Gordy (2003), for asymptotically fine-grained sector portfolios, the stand-alone α-percentile portfolio loss for a

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[7] For consistency with Basel II, we focus on a one-period Merton model for default losses. The methodology and results are quite general and can be used with other credit models, and can also incorporate losses due to credit migration, in addition to default.

given sector *k*, $VaR_k(\alpha)$, is given by the sum of the individual obligor losses in that sector, when an α-percentile move occurs in the systemic sector factor Z_k:

$$VaR_k(\alpha) = \sum_{j \in Sector\ k} LGD_j \cdot EAD_j \cdot N\left(\frac{N^{-1}(PD_j) - \sqrt{\rho_k}\ z^\alpha}{\sqrt{1-\rho_k}}\right)$$

where z^α denotes the α-percentile of a standard normal variable.

Consistent with common risk practices and with the Basel II capital rule, we define the *stand-alone capital* for each sector, $C_k(\alpha)$, to cover only the *unexpected losses*. Thus, $EC_k(\alpha) = VaR_k(\alpha) - EL_k$, where $EL_k = \sum_{j \in Sector\ k} LGD_j \cdot EAD_j \cdot PD_j$ are the expected sector losses.[8] The capital for sector *k* can then be written as

$$EC_k(\alpha) = \sum_{j \in Sector\ k} LGD_j \cdot EAD_j \cdot \left[N\left(\frac{N^{-1}(PD_j) - \sqrt{\rho_k}\ z^\alpha}{\sqrt{1-\rho_k}}\right) - PD_j \right] \quad (5)$$

Under Basel II, or equivalently assuming perfect correlation between all the sectors, the overall capital is simply the sum of the stand-alone capital for all individual sectors

$$EC^{sf} = \sum_{k=1}^{K} EC_k \quad (6)$$

(for simplicity, we omit the parameter α hereafter).

The Diversification Factor and Capital Diversification Index

In equation (1), we define the *diversification factor*, *DF,* as the ratio of the capital computed using the multi-factor model and the stand-alone capital (now defined in equation 6), $DF = EC^{mf} / EC^{sf}$, $DF \leq 1$.

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[8] The following discussion still holds if capital is defined by VaR, by simply adding back the *EL* at the end of the analysis.

As given in equation (2), for a given quantile, we seek to approximate *DF*, by a scalar function of a small number of intuitive parameters (say two or three). This allows us to express the (diversified) economic capital as a function of the "additive" bottom-up capital from the one-factor model (equation 6), and a "factor adjustment" (which can be tabulated)

$$EC^{mf} \approx DF(\cdot) \times \sum_{k=1}^{K} EC_k$$

Let us now first motivate the parameters used for this approximation. We can think of diversification basically being a result of to two sources. The first one is the correlation between the sectors. Hence, a natural choice for a parameter in our model is the correlation β of the systemic sector factors Z_k. The second source is the relative size of various sector portfolios. Clearly, one dominating very large sector leads to high concentration risk and limited diversification. So we seek a parameter representing essentially an "effective number of sectors" accounting for their sizes. Ideally, this should also account for the differences in credit characteristics as they affect capital. Thus, a sector with a very large exposure on highly rated obligors, might not necessarily represent a large contribution from a capital perspective.

Define the *capital diversification index*, *CDI*, as the sum of squares of the *capital weights* in each sector

$$CDI = \frac{\sum_k {EC_k}^2}{\left(EC^{sf}\right)^2} = \sum_k {w_k}^2 \tag{7}$$

with $w_k = EC_k / EC^{sf}$ the contribution to one-factor capital of sector *k*. The *CDI* is simply the well-known Herfindahl concentration index applied to the stand-alone capital of each sector (rather than to the exposures, as is more commonly used). Intuitively, it gives an indication of the portfolio diversification across sectors (not accounting for the correlation between them). For example, in the two-factor case, the *CDI* ranges between 0.5 (maximum diversification) and one (maximum concentration). The inverse of the *CDI* can be interpreted as an "effective number of sectors" in the portfolio, from a capital perspective. Note that one can similarly define the Herfindahl index for sector or counterparty exposures (*EADs)*, which results in a measure of concentration in terms of the size of the portfolio (and not necessarily the capital).

It is easy to understand the motivation for introducing the *CDI*. For a set of uncorrelated sectors, the standard deviation of the overall portfolio loss distribution is given by $\sigma_P = \sqrt{CDI} \sum_k \sigma_k$, with σ_P, σ_k the volatilities of credit losses for the portfolio and sector *k*, respectively. More generally, for correlated sectors, denote by $\tilde{\beta}$ the single correlation parameter of credit losses (and not the asset correlations). Then, the volatility of portfolio credit losses is given by[9]

$$\sigma_P = \sqrt{\left(1-\tilde{\beta}\right)CDI + \tilde{\beta}} \sum_k \sigma_k \tag{8}$$

If credit losses were normally distributed, a similar equation to (8) would apply for the credit capital at a given confidence level, $EC^{mf} = DF^N(CDI, \tilde{\beta}) \cdot EC^{1f}$, with $DF^N = \sqrt{\left(1-\tilde{\beta}\right)CDI + \tilde{\beta}}$, the diversification factor for a Normal loss distribution. Figure 1 shows a plot of DF^N as a function of the *CDI* for different levels of the sector loss correlation, $\tilde{\beta}$. For example, for a *CDI* of 0.2 and a correlation of 25%, the diversified capital from a multi-factor model is about 60% of the one-factor capital, if the distribution is close to Normal).

Although credit loss distributions are not Normal, it seems natural to attempt a two-factor parameterization for equation (1) such as

$$EC^{mf}\left(CDI, \beta\right) \approx DF\left(CDI, \beta\right) \cdot EC^{sf} \tag{9}$$

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[9] One can explicitly obtain the relationship between asset and loss correlations. For the simplest case of large homogeneous portfolios of unit exposures, default probability, *PD*, with a single intra-sector asset correlation ρ and correlation of sector systemic factors β, the systemic credit loss correlation is given by

$$\tilde{\beta} = \left[N_2\left(N^{-1}(PD), N^{-1}(PD), \rho\beta\right) - PD^2\right] / \left[N_2\left(N^{-1}(PD), N^{-1}(PD), \rho\right) - PD^2\right]$$

with $N_2(a,b,\rho)$ the standard bivariate normal distribution of random variables *a* and *b* and correlation ρ. Note also that the variance of portfolio losses is given by the well-known formula

$$\sigma_p^2 = \sum_{i,j} LGD_i EAD_i LGD_j EAD_j \left[N_2\left(N^{-1}(PD_i), N^{-1}(PD_j), \rho_{ij}\right) - N^{-1}(PD_i) N^{-1}(PD_j)\right]$$

where $\rho_{ij} = \rho_k$ for obligors in the same sector and $\rho_{ij} = \beta\sqrt{\rho_k}\sqrt{\rho_l}$ for obligors in different sectors.

with the sector systemic factor correlation substituting the loss correlation, given it's availability, *a priori*, from the underlying model. In the rest of the paper, we refer to the model given by equations (3), (4), (5), (6) and (9) as the *DF credit capital model*.

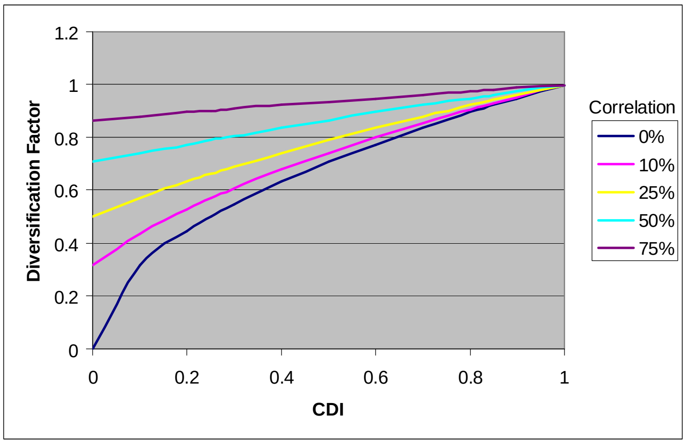

Figure 1. Idealized diversification factor for Normal distributions

Clearly, we do not expect the parameterization (9) to be exact, nor for the *DF* to follow necessarily the same functional form as DF^N. However, as explained earlier, we can expect the two parameters to capture broadly the key sources for diversification: homogeneity of sector sizes and cross-sector correlation. So it remains an empirical question to see whether these two parameters are enough to create a reasonable approximation of the diversification factor. Note also that, for regulatory use, we might seek to estimate a *conservative* diversification factor *DF*, so finding a reasonable upper bound might be more appropriate for this type of application.

Estimating the Diversification Factor, *DF*

We propose to estimate the *DF* function numerically using Monte Carlo simulations. In general, this exercise requires the use of a multi-factor credit portfolio application (which might itself use a simulation technique). The parameterization obtained for *DF* can then be tabulated and used generally both as a basis for minimum capital requirements and for quick approximations of economic capital in a multi-factor setting, without recourse to further simulation.

The general parameterization methodology is as follows. We assume in each simulation, a set of homogeneous portfolios representing each sector. Each sector is assumed to contain an infinite number of obligors with the same *PD* and *EAD*. Without loss of generality, we set *LGD* = 100%, and the total portfolio exposure equal to one, $\sum EAD_k = 1$.

The numerical experiments are performed as follows:

- Assume a fixed average cross-sector correlation β and number of sectors *K*. We run a large number of capital calculations, varying independently in each experiment[10]:
  - the sizes of each sector
  - PD_k , EAD_k , ρ_k , $k = 1,...,K$
- In each run, we compute EC_k $(k = 1,...,K)$, EC^{sf} and *CDI* from the simple one-factor analytical formula and also the "true" EC^{mf} from a full multi-factor model[11]
- We plot the ratio of (EC^{mf} / EC^{sf}) vs. the *CDI*.
- To get the overall *DF* function for a level of correlation β we then repeat the exercise varying the number of sectors *K*
- We then repeat the exercise for various levels of correlation
- Finally, we estimate the function *DF* (*CDI*, β) by fitting a parametric function to the points

As an example, Figure 2 presents the plot for *K*=2 to 5 and β=25% and random independent draws with $PD_k \in [.02\%,20\%]$, $\rho_k \in [2\%,20\%]$. The dots represent the various experiments, each with different parameters. The colours of the points represent the different number of sectors. Simply for reference, for each *K*, we also plot the convex polygons enveloping the points. Figure 2 shows that the approximation is not perfect, otherwise all the points would lie on a line (not necessarily straight). However, all the points do lie within a well bounded area, suggesting it as a reasonable approach. A function *DF* can be reliably parameterized either as a fit to the points or, more conservatively, as their envelope. For example, for a *CDI* of 0.5, a diversification factor of 80% results in a conservative estimate of the capital reduction incurred by diversification.

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[10] In practice, one must use reasonable ranges for the parameters as required by the portfolio. For Basel II adjustments, we do not have to sample independently the asset correlations ρ_k , since these are either constant or prescribed functions of *PD*, for each asset class. As shown later, this results in tighter estimates.

[11]Except for the two-factor case, where numerical integration can be used, multi-factor capital is calculated using a MC simulation, although some analytics might be possible as explained earlier.

This exercise is only meant to illustrate the parameterization methodology. We have shown that even in the case where sector *PD*s, exposures and intra-sector correlations are varied independently, two factors (*CDI*, β) provide a reasonable explanation of the diversification factor. One can get tighter approximations by adding explanatory variables or by constraining the set over which the approximation is valid. In practice, for example, *PDs* and intra-sector correlations do not vary independently and they might only cover a smaller range. In Section 4, we provide a more rigorous parameterization and examples in the context of the Basel II formulae.

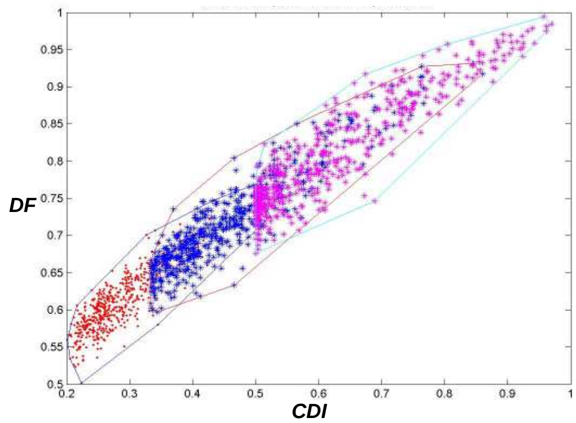

Figure 2. Empirical *DF* as a function of the *CDI* (*K*=2 to 5, and β=25%)

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Capital Allocation and Risk Contributions

Under a one-factor credit model, capital allocation is straightforward. The capital attributed to a given sector is the same as its stand-alone capital, EC_k, since the model does not allow further diversification. Under the full multi-factor model, the total capital is not necessarily the sum of the stand-alone capitals in each sector. Clearly, the standalone risk of each component does not represent a valid contribution for sub-additive risk measures in general, since it fails to reflect the beneficial effects of diversification. Rather, it is necessary to compute contributions on a marginal basis. The theory behind marginal risk contributions and additive capital allocation is well developed and the reader is referred elsewhere for its more formal derivation and justification (e.g. Gouriéroux et al 2000, Hallerbach 2003, Kurth and Tasche, 2003, Kalkbrener et al 2004).

Using the factor adjustment approximation (9), one might be tempted simply to allocate back the diversification effect evenly across sectors, so that the total capital contributed by a given sector is $DF \cdot EC_k$. We refer to these as the *unadjusted capital contributions*. This would not account, however, for the fact that each sector contributes differently to the overall portfolio diversification. Instead, we seek a capital decomposition of the form

$$EC^{mf} = \sum_{k=1}^{K} DF_k \cdot EC_k \tag{10}$$

We refer to the factors DF_k in equation (10) as the *marginal sector diversification factors*.

If *DF* only depends on *CDI* and β (where the correlation can also represent an average correlation for all sectors, as shown below), it is then a homogeneous function of degree zero in the EC_k's (indeed it is homogeneous in the size of each sector exposures as well). This is a direct consequence of both the *CDI* and the average β (as defined later) being homogenous of degree zero. Thus, the multi-factor capital formula (9) is a homogeneous function of degree one. Applying Euler's theorem, leads to the additive marginal capital decomposition (10) with[12]

$$DF_k = \frac{\partial EC^{mf}}{\partial EC_k} \quad , \quad k = 1,\dots,K \tag{11}$$

Under the simplest assumption that all sectors have the same correlation parameter β, we can show that

$$DF_k = DF + 2DF' \cdot \left[\frac{EC_k}{EC^{sf}} - CDI \right] \tag{12}$$

where $DF' = \partial DF / \partial CDI$ is the slope of the factor adjustment for the given correlation level β. Expression (11) shows that the marginal sector diversification factor is a combination of the overall portfolio *DF* plus an adjustment due to the *"relative size"* of the sector to the overall portfolio. Intuitively, for *DF* >0 and all sectors having the same correlation β, a sector with small stand-alone capital ($EC_k / EC^{sf} < CDI$) contributes, on the margin, less to the overall portfolio capital; thus, it gets a higher diversification benefit DF_k.

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[12] Tasche (2006) formally generalizes the diversification factor and the marginal diversification factors introduced here for a general risk measure (e.g. he defines the marginal diversification factor of a given position, with respect to a given risk measure, as the ratio of its risk contribution and its stand alone risk).

In the more general case, each sector has a different correlation level β_k. We define in general the average factor correlation as follows.

Assume a general sector factor correlation matrix, Q (this can be more general than that resulting from equation 2, where $Q_{ij} = \sqrt{\beta_i \beta_j}$, $j \neq i$), and a vector of portfolio weights $W = (w_1 ... w_S)^T$.
We define the average sector factor correlation as

$$\overline{\beta} = \frac{\sum_i \sum_{j \neq i} Q_{ij} w_i w_j}{\sum_i \sum_{j \neq i} w_i w_j} = \frac{\sigma^2 - \delta^2}{\vartheta^2 - \delta^2}$$

where $\sigma^2 = W^T Q W$ is the variance of the random variable given by the weighted sum of the factors, $\delta^2 = \sum_i w_i^2$ and $\vartheta^2 = \left(\sum_i w_i\right)^2$. $\overline{\beta}$ is an average correlation in the sense that $W^T B W = W^T Q W = \sigma^2$, with B the correlation matrix all the non-diagonal entries equal to $\overline{\beta}$. For our specific case, we chose the portfolio weights to be the stand alone capital for each sector. Therefore, $\delta^2 = \sum_i EC_i^2$ and $\vartheta^2 = \left(\sum_i EC_i\right)^2 = \left(EC^{sf}\right)^2$.

Then, the marginal sector diversification factor is given by

$$DF_k = DF + 2\frac{\partial DF}{\partial CDI} \cdot \left[\frac{EC_k}{EC^{sf}} - CDI\right] + 2\frac{\partial DF}{\partial \overline{\beta}} \cdot \frac{1 - \left(EC_k / EC^{sf}\right)}{1 - CDI} \cdot \left[\overline{Q}_k - \overline{\beta}\right] \tag{13}$$

where

$$\overline{Q}_k = \frac{\sum_{j \neq k} Q_{kj} EC_j}{\sum_{j \neq k} EC_j}$$

is the average correlation of sector factor k to the rest of the systemic sector factors in the portfolio. Thus, sectors with lower than average correlation to the rest of the systemic sector factors in the portfolio get a higher diversification benefit, as one would expect.

The marginal capital allocation resulting from the model leads to an intuitive decomposition of diversification effects (or concentration risk) into three components: overall portfolio diversification, sector size and sector correlation:[13]

$$DF_k = DF + \Delta DF_{Size} + \Delta DF_{Corr} \tag{14}$$

4. Parameterization Exercises

Section 3 presented a simple example to illustrate the parameterization methodology for a general problem where sector *PDs*, exposures and intra-sector correlations where varied independently. Even in this case, two parameters (*CDI*, β) provided a reasonable explanation of the diversification factor. One can get a tighter approximation, by either searching for more explanatory variables, or by constraining the set over which the approximation is valid. In practice, *PDs* and intra-sector correlations do not vary independently and they might only vary over smaller ranges. For example, under the Basel II capital rules, the asset correlation is either constant on a given asset class (e.g. revolving retail exposures, at 4%) or varies as a function of *PDs* (e.g. wholesale exposures).[14] See also Lopez (2004), which shows that average asset correlation is a decreasing function of *PD* and an increasing function of asset size.

In this section, we present more rigorous parameterizations and error analysis for the case of wholesale exposures (corporates, banks and sovereign) in the context of Basel II. We first describe in detail the case of a two-factor parameterization and a given cross-sector correlation β, and then extend the results further to multiple factors and correlation levels. Our objective in this section is not to provide a complete parameterized surface, but rather to develop a good understanding of the basic characteristics of the diversification factor surface, the approximation errors and the robustness of the results.

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[13] When one defines the average correlation as an arithmetic average, $\overline{\beta} = \sum (EC_k / EC^{sf}) \cdot \beta_k$, the resulting formula for the marginal sector diversification factor is simpler and given by

$$DF_k = DF + 2\frac{\partial DF}{\partial CDI} \cdot \left[\frac{EC_k}{EC^{sf}} - CDI\right] + \frac{\partial DF}{\partial \overline{\beta}} \cdot \left[\beta_k - \overline{\beta}\right]$$

Although simpler, this definition has some undesirable properties which result in inconsistencies.

[14] In this case, the asset correlation is given by

$$\rho = 0.12\left(\frac{1-e^{-50PD}}{1-e^{-50}}\right) + 0.24\left(1 - \frac{1-e^{-50PD}}{1-e^{-50}}\right)$$

Two-Dimensional Parameterization for Wholesale Exposures

Consider a portfolio of wholesale exposures in two homogeneous sectors, each driven by a single factor model. We assume a cross-sector correlation β = *60%*. For simplicity, assume all loans in the portfolio have a maturity of one year. To estimate the diversification factor function, *DF* (*CDI*, β=*60%*), we perform a Monte Carlo simulation of three thousand portfolios. The *PDs* for each sector portfolio are sampled randomly and independently, from a uniform distribution in the range [0,10%]. We further assume that in each sector, asset correlations are given as a function of *PD*s from the Basel II formula for wholesale exposures without the firm-size adjustment. The percent exposure in each sector is sampled randomly as well, and without loss of generality we assume 100% *LGDs.* For each of the 3,000 portfolios, the economic capital is calculated using a MC simulation with one million scenarios on the sector factors (assuming β=*60%*), and assuming these are granular portfolios (hence computing the conditional expected portfolio losses under each scenario). Economic capital is estimated as the 99.9% percentile of the credit losses net of the expected losses.

Figure 3 compares the capital obtained for the simulated portfolios using a one-factor model and a two-factor model, as a function of the average default probability (to make the number more realistic, we plot the capital assuming 50% *LGD*s). The two-factor model generally results in capital requirements that are lower than those of the single-factor model, as the circles (in blue), which correspond to the single-factor model, are generally above the squares (in red), which correspond to the two-factor model.

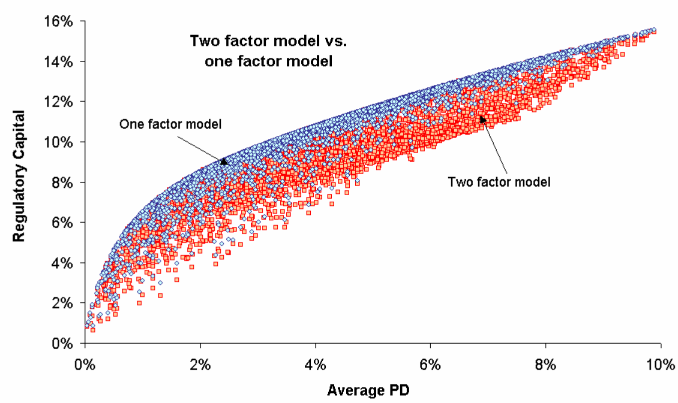

Figure 3. One-factor and two-factor capital as a function of average *PD*s (*LGD=50%*)

Figure 4 plots the diversification factor, *DF*, as a function of the *CDI* for the simulated portfolios. With two factors, the *CDI* ranges between 0.5 (maximum diversification) and 1 (maximum concentration). There is a clear relationship between the diversification factor and the *CDI*, and a simple linear model fits the data very well, with an R^2 of 0.96. Thus, we can express the diversification factor as[15]

$$DF(CDI, \beta = 0.6) = 0.6798 + 0.3228 \cdot CDI$$

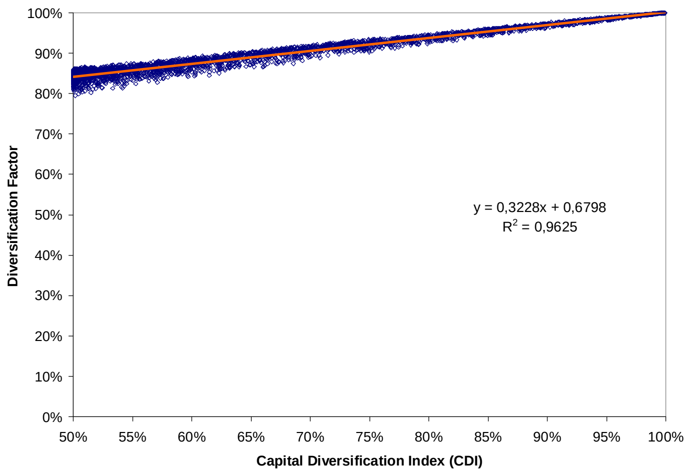

Figure 4. Two-factor diversification factor as a function of the *CDI* (β=60%)

Figure 5 displays, for all simulated portfolios, the actual economic capital from the two-factor model against that estimated from the *DF* model resulting from the regression in Figure 4. There is clearly a close fit between the two models, with the standard error of the estimated diversification factor model of only 10 basis points. Finally, Table 5 summarizes the resulting diversification factor in table format. Accounting for maximum diversification, the capital savings are 16% .

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[15] Similarly, one can obtain the parametric envelop of the data, to get a more conservative adjustment.

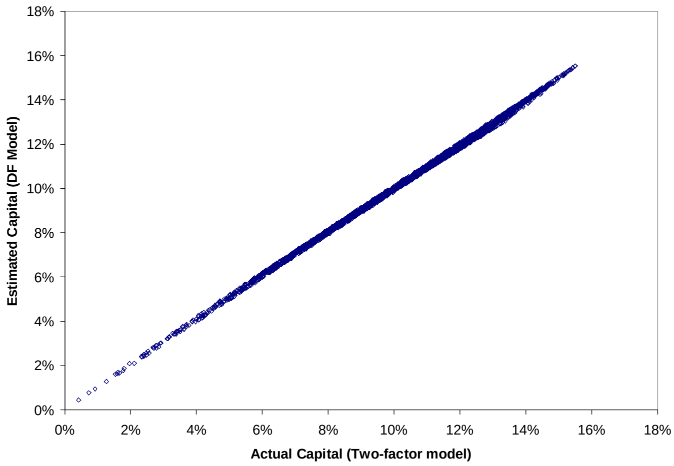

Figure 5. Capital from *DF* model vs. actual two-factor capital (β=60%)

CDI	Diversification Factor
50%	84%
55%	86%
60%	87%
65%	89%
70%	91%
75%	92%
80%	94%
85%	95%
90%	97%
95%	99%
100%	100%
Intercept	0.6798
slope	0.3228
R^2	0.97

Table 5. Tabulated diversification factor (two-factors) (β = 60%)

To understand the application of this resulting model to capital allocation, consider a portfolio with 70% of the one-factor capital in sub-portfolio 1 and 30% in sub-portfolio 2. Table 6 presents a summary of the capital contributions. The CDI = 0.58, which leads to DF = 86.3% . As defined earlier, the unadjusted capital contributions apply the same diversification factor of 86.3% to each sub-portfolio, thus retaining the same proportion of allocation as the SA contributions. However, consistent with a marginal risk allocation, the smaller portfolio contributes more to the overall diversification and gets an adjustment factor of 67%, while the larger portfolio gets a 94% factor. The marginal capital contributions of the portfolios are 66.1 (76.6%) and 20.2 (23.4%), respectively (summing to 86.3).

	Capital One-Factor	SA Capital Contributions %	Unadjusted Capital Contributions	Marginal Sector Diverisfication Factor	Marginal Sector Capital Contributions	Marginal Sector Capital Contributions %
P1	70.0	70.0%	60.4	**0.94**	66.1	**76.6%**
P2	30.0	30.0%	25.9	**0.67**	20.2	**23.4%**
Total	100.0	100%	86.3		86.3	**100%**
		CDI	**0.58**			
		DF	**86.3%**			

Table 6. Capital contributions for a two-factor model (β=60%)

Parameterization of the Surface

We now investigate the behaviour of the surface as a function of the number of factors and also for other cross-sector correlation levels. We now consider portfolios of wholesale exposures consisting of k homogeneous sectors, $k=2,3,\ldots,10$. The cross-sector correlation is $\beta = 60\%$. We follow the same estimation procedure as before to estimate the diversification factor function, $DF(CDI, \beta=60\%)$ for each k, using Monte Carlo simulations of three thousand portfolios, each.

Figure 6 shows the detailed regression plots for $k=4, 7, 10$. Table 7 presents the *DF* tabulated for each k. It also presents the coefficients of the regressions and, finally, an average over all the range. In all cases from 2-10 factors linear model fits the data well with R^2 ranging from 96-98%, and standard approximation errors of 10-11 bps. It is clear that at this correlation level, a linear model fits the data very well, from this example, as is further shown in Figure 7, which plots the nine regression lines.

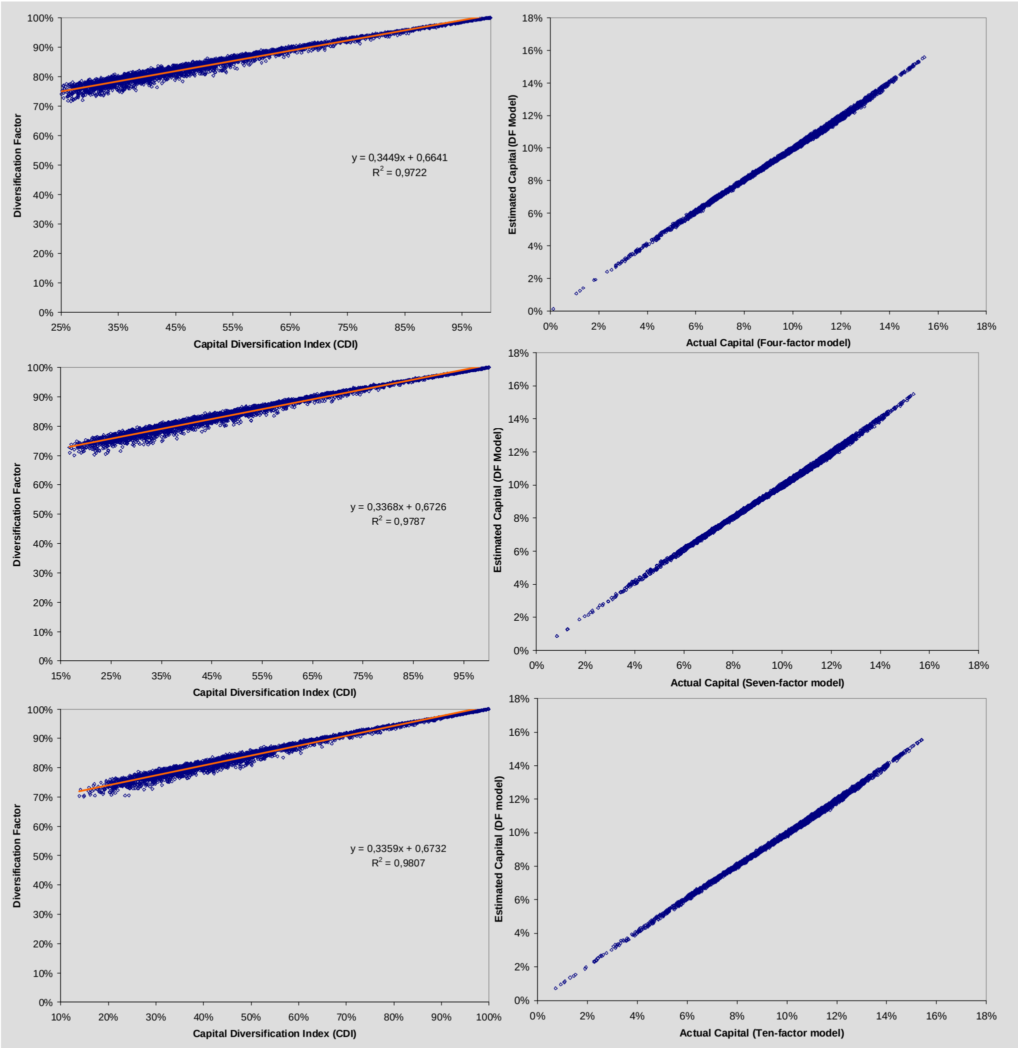

Figure 6. ***DF*** **model regressions for** ***k=4, 7, 10*** **(β=60%)**

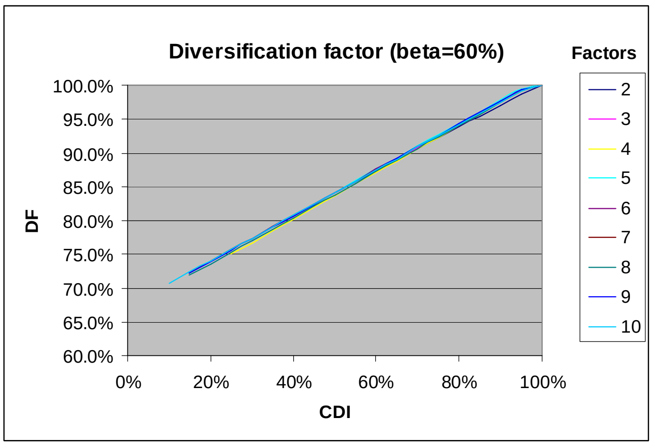

Figure 7. ***DF*** **model regression lines for** ***k=2, …, 10*** **(β=60%)**

CDI \ Factors	2	3	4	5	6	7	8	9	10	Average
10%									70.7%	70.7%
15%						72.3%	71.9%	72.2%	72.4%	72.2%
20%				74.0%	74.0%	74.0%	73.6%	73.9%	74.0%	73.9%
25%			75.0%	75.7%	75.7%	75.7%	75.3%	75.6%	75.7%	75.5%
30%			76.8%	77.4%	77.4%	77.4%	77.0%	77.3%	77.4%	77.2%
35%		79.1%	78.5%	79.1%	79.1%	79.0%	78.7%	79.0%	79.1%	78.9%
40%		80.7%	80.2%	80.8%	80.8%	80.7%	80.4%	80.7%	80.8%	80.6%
45%		82.4%	81.9%	82.5%	82.5%	82.4%	82.1%	82.4%	82.4%	82.3%
50%	84.1%	84.1%	83.7%	84.2%	84.2%	84.1%	83.8%	84.1%	84.1%	84.0%
55%	85.7%	85.8%	85.4%	85.9%	85.8%	85.8%	85.5%	85.8%	85.8%	85.7%
60%	87.3%	87.4%	87.1%	87.6%	87.5%	87.5%	87.2%	87.5%	87.5%	87.4%
65%	89.0%	89.1%	88.8%	89.3%	89.2%	89.2%	88.9%	89.2%	89.1%	89.1%
70%	90.6%	90.8%	90.6%	91.0%	90.9%	90.8%	90.6%	90.9%	90.8%	90.8%
75%	92.2%	92.5%	92.3%	92.7%	92.6%	92.5%	92.4%	92.6%	92.5%	92.5%
80%	93.8%	94.1%	94.0%	94.4%	94.3%	94.2%	94.1%	94.3%	94.2%	94.1%
85%	95.4%	95.8%	95.7%	96.1%	95.9%	95.9%	95.8%	96.0%	95.9%	95.8%
90%	97.0%	97.5%	97.5%	97.8%	97.6%	97.6%	97.5%	97.7%	97.5%	97.5%
95%	98.6%	99.1%	99.2%	99.5%	99.3%	99.3%	99.2%	99.4%	99.2%	99.2%
100%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Intercept	0.6798	0.6734	0.6641	0.6722	0.6731	0.6726	0.6675	0.6706	0.6732	0.6718
slope	0.3228	0.3349	0.3449	0.3397	0.3369	0.3368	0.3413	0.3406	0.3359	0.3371
R^2	96.3%	96.9%	97.2%	97.6%	98.0%	97.9%	97.9%	98.0%	98.1%	

Table 7. Tabulated results for the *DF* model for *k=2,…10* (β=60%)

Figure 8 plots the linear regressions from the same exercise for a correlation of β=40%, for *k=2,…,10*. The R^2 are in the order 97 to 98% and the standard errors range between 12-15 bps.

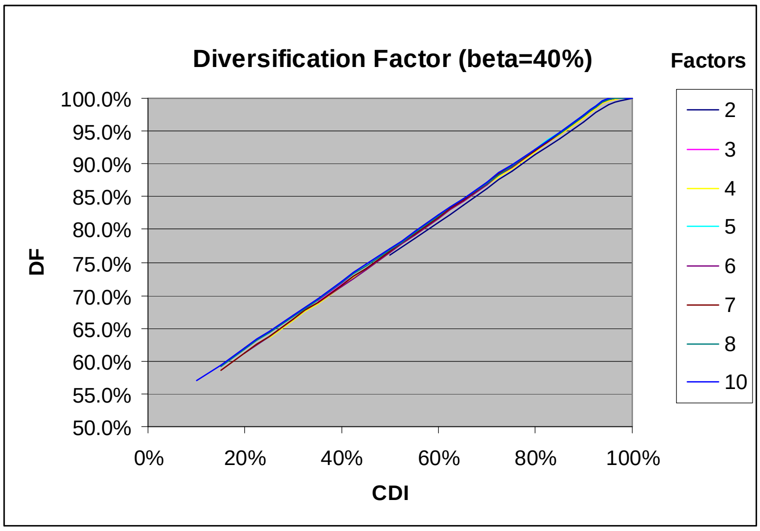

Figure 8. *DF* regression lines for *k=2, …, 10* (β=40%)

A linear regression still performs quite well in fitting the actual economic capital for the MC generated portfolios, but is not as accurate as in the previous case (β=60%). The effect of curvature is illustrated in Figure 9, which shows a linear and a quadratic fit through the data for the case when the portfolio contains 10 sectors.

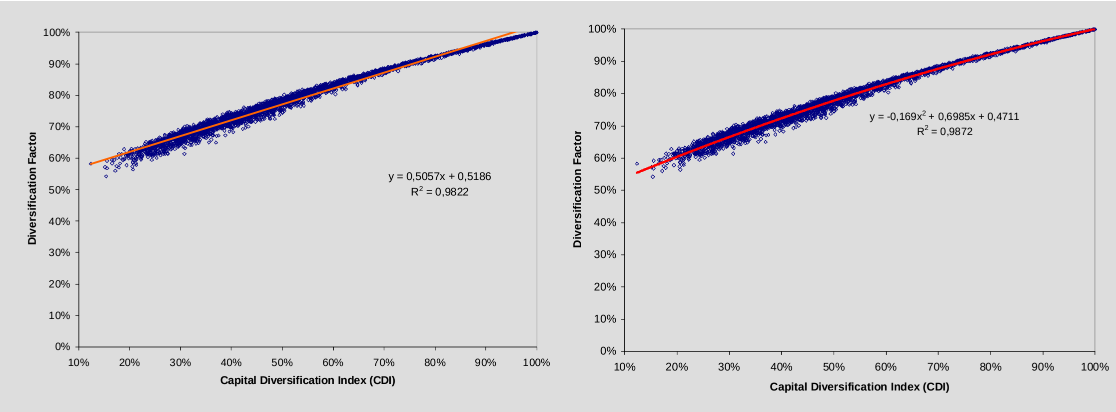

Figure 9. ***DF*** **model linear and quadratic fit for** ***k=10*** **(β=40%)**

The quadratic fit clearly fits the data better, and in particular at both ends of the range, where the linear fit is clearly off (e.g. resulting in a higher than 100% diversification factor, which would need to be capped). Figure 10 plots the average linear and quadratic fits and provides the functions in tabular form for comparison. There are differences in the estimated *DF* of up to 3%. In practice, the quadratic fit provides added value. This quadratic model is given by

CDI	Linear	Quadratic
10%	56.9%	53.9%
15%	59.1%	57.0%
20%	61.5%	59.9%
25%	63.9%	62.8%
30%	66.5%	65.9%
35%	69.1%	68.8%
40%	71.6%	71.8%
45%	74.2%	74.6%
50%	76.7%	77.1%
55%	79.2%	79.9%
60%	81.8%	82.5%
65%	84.3%	85.0%
70%	86.9%	87.5%
75%	89.5%	89.8%
80%	92.0%	92.1%
85%	94.6%	94.2%
90%	97.1%	96.2%
95%	99.7%	98.2%
100%	100.0%	100.0%

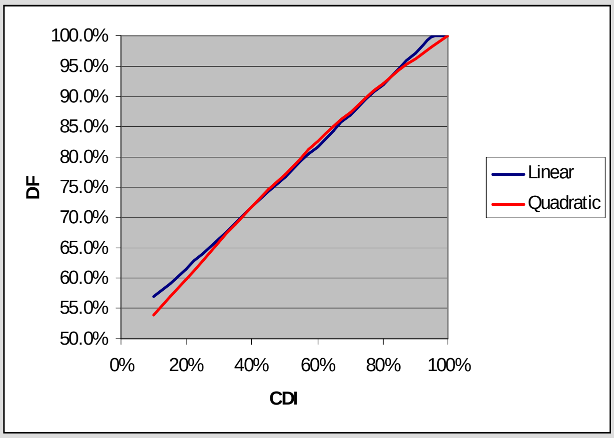

Figure 10. ***DF*** **model linear and quadratic functions (β=40%)**

The non-linear nature of the *DF* tends to increase with decreasing correlation level. One can get some intuition to this by revisiting the functional form for portfolio loss standard deviation as given by equation (8) and Figure 1. To illustrate this effect further, Figures 10 and 11 present he results for two uncorrelated factors (β=0%). [16]

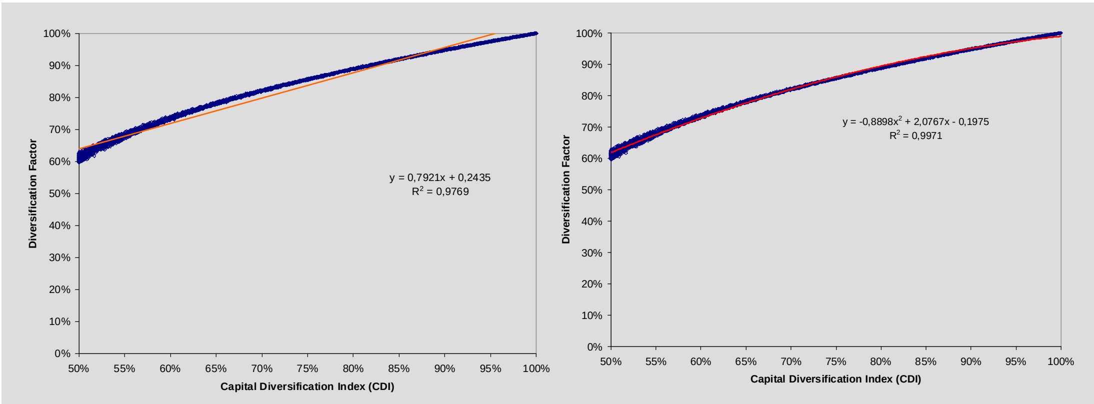

Figure 11. *DF* model linear and quadratic fit for *k=2* (β=0%)

CDI	Linear	Quadratic
50%	64.0%	61.8%
55%	67.9%	67.6%
60%	71.9%	72.8%
65%	75.8%	77.6%
70%	79.8%	82.0%
75%	83.8%	86.0%
80%	87.7%	89.4%
85%	91.7%	92.5%
90%	95.6%	95.1%
95%	99.6%	97.2%
100%	100.0%	100.0%

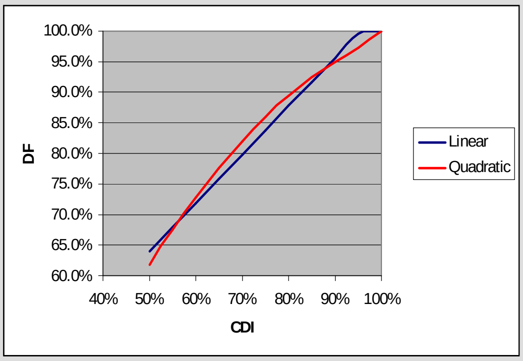

Figure 12. *DF* model linear and quadratic functions (β=0%)

Finally, to get an overall picture of the *DF* surface, Figure 13 plots the function for the three levels of correlation, as computed in this section. Note the similarity of with Figure 1.

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[16] In Figure 12, the *DF* is capped at 100% and also the quadratic function is adjusted at the end to get precisely *DF*=100% for a 100% CDI.

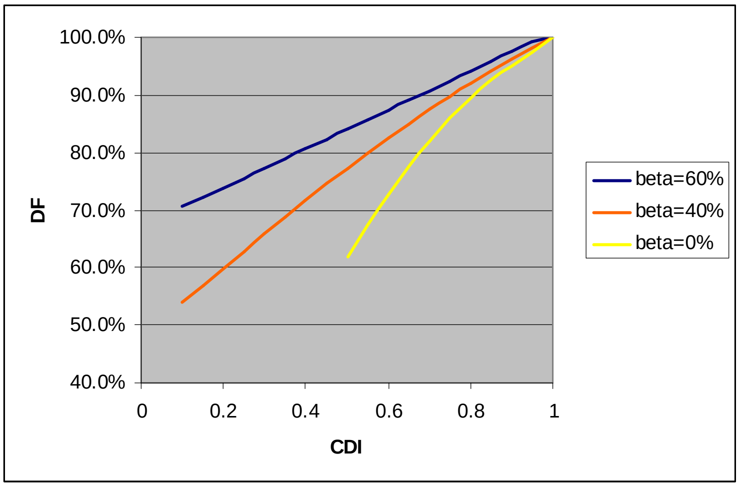

Figure 13. ***DF*** **model linear and quadratic functions (*β*=0%)**

5. The Diversification Factor as a management tool

In addition to its potential regulatory applications, we now focus on the application of the *DF* model as a risk management tool to

- understand concentration risk and capital allocation
- identify capital sensitivities to sector size and correlations
- compute "real-time" marginal risk contributions for new deals or portfolios

In this section, we first summarize the parameters of the model and the sensitivities derived from it, and discuss their interpretation as risk and concentration indicators. We then explain how the model can be used in conjunction with a full multi-factor internal credit capital model, by computing its implied parameters. We illustrate this application with a simple example.

Summary of Model Parameters as Risk and Concentration Indicators

The intuitiveness of the *DF* model allows us to view its parameters as useful risk and concentration summary indicators. We divide these into, *sector-specific indicators, portfolio capital indicators, capital contributions and correlations, and sensitivities*. For completeness, we summarize these in Table 8.

Sector specific indicators (for sectors k=1,…,K) [17]		***Portfolio capital indicators***		***Marginal capital contributions (for sectors k=1,…,K)***	
Inputs					
ρ_k	Intra-sector (asset) correlation	EC^{sf}	Capital one-factor (undiversified)	β_k	Sector factor correlation weights
$\overline{PD}_k$	average default probability	*CDI*	Capital diversification index	$\overline{Q}_k$	Average correlation of a sector factor to the other sectors
$\overline{EAD}_k$ $\overline{LGD}_k$	Average exposure, loss given default	$\overline{\beta}$	Average cross sector correlation		
Outputs					
EC_k	Stand-alone capital	*DF*	Diversification factor	DF_k	Sector diversification factor $DF_k = DF + \Delta DF_k^{size} + \Delta DF_k^{corr}$
		EC^{mf}	Economic capital (diversified)	ΔDF_k^{size}	Sector size diversification component
		$\frac{\partial DF}{\partial \overline{\beta}}$	Sensitivity of *DF* to changes in average cross-correlation	ΔDF_k^{corr}	Sector's correlation diversification component
		$\frac{\partial DF}{\partial CDI}$	Sensitivity of *DF* to changes in *CD*		

Table 8. Summary parameters and risk indicators of *DF* model

We obtain the sensitivities of the diversification factor to the *CDI* and the average cross-sector correlation directly as slopes from the estimated *DF* surface. By using the chain rule, it is straightforward to get the sensitivities of the factor to the sector SA capital (EC_k) or to its correlation parameters ($\overline{Q}_k$, β_k). In addition, the following sensitivities are useful for management purposes:

- $\partial EC^{mf} / \partial EC_k = DF_k$, $(k = 1,..., K)$ – change in economic capital per unit of stand-alone capital for *k*-th sector (it can also be normalized on a per unit exposure basis)

[17] Commonly, the (exposure-weighted) average *EAD* and *LGD* for each sector are computed, and the average *PD is* implied from the actual calculation of expected losses.

- $\partial EC^{mf} \big/ \partial \overline{\beta} = df^{c} \cdot EC^{sf}$ – change in economic capital per one unit of average correlation (with $df^{c} = \partial DF / \partial \overline{\beta}$, as above, the slope of the DF surface in the direction of the average correlation)
- $\partial EC^{mf} \big/ \partial \beta_k = \left(\partial EC^{mf} \big/ \partial \overline{\beta} \right) \cdot \left(\partial \overline{\beta} \big/ \partial \beta_k \right) = df^{c} \cdot \left[EC^{sf} \Big/ \left(\left(EC^{sf} \right)^2 - \delta^2 \right) \right] \cdot \left(\sum_{i \neq k} \sqrt{\beta_i} \Big/ \sqrt{\beta_k} \right), (k = 1, \ldots, K)$ – change in economic capital per one unit of sector factor correlation for k-th sector

Implied Parameters for an Internal Multi-Factor Economic Capital Model

The *DF* model can be fitted effectively to a full multi-factor economic capital model by calculating its implied parameters. The fitted model, with its implied parameters, then can be used to understand the underlying problem better, for communication purposes, or as a simpler and much faster model for real-time calculation or extrapolation. In this sense, this is akin to using the implied volatility surface from option prices with the Black-Scholes model, or the implied correlation skew in CDOs in the context of a copula model.

Assume, for ease of exposition, that we have divided the portfolio into *K* homogeneous sectors (not necessarily granular), each with a single *PD*, *EAD* and *LGD* (in practice this latter assumption can be relaxed).[18] The inverse problem solves for *2K* implied correlation parameters (ρ_k, β_k), thus requiring as many statistics from the internal model. A straightforward algorithm to fit the model is as follows:

- Compute for each sector portfolio $k=1,\ldots,K$, its stand-alone capital from the internal multi-factor economic capital model
- Solve for the implied intra-sector correlation, ρ_k , from equation (5). If the portfolio is fully granular (or we are simply interested in systemic capital), this provides an indication of the average correlation (even for non- homogeneous portfolios). For non-granular portfolios, this

[18] Sector homogeneity is not a requirement. Note that equation (4) does not require single *PDs*, *EADs* and *LGDs* for each sector.

implied correlation adjusts the model granularity effects; the less granular the portfolio, the higher the implied correlation.[19]

- Compute the total stand-alone capital, EC^{sf}, and *CDI* from the *K* stand-alone capitals EC_k for each sector.
- Compute the overall economic capital for the portfolio, EC^{mf}, from the internal multi-factor capital model.
- Solve for the average correlation, $\overline{\beta}$, implied from the equation (9)

$$EC^{mf}\left(CDI,\beta\right) = DF\left(CDI,\overline{\beta}\right)\cdot EC^{sf} \;,$$

  assuming that the *DF* surface is available in parametric (or non-parametric) form
- Computes the *K* marginal capital contributions to each sector, $DF_k \cdot EC_k$, from the internal economic capital model.
- Solve for the implied inter-sector correlation parameters $\overline{Q}_k$ and β_k from the marginal capital contributions.

We can see from this algorithm, that the *DF* model basically provides a map from the correlation parameters to various capital measures:

- intra-sector correlations ←→ stand-alone capital
- overall capital (or the diversification factor) ←→ average cross-sector correlation
- marginal capital contributions ←→ relative sector size and relative cross-sector correlation

Example: Model with Implied Parameters

We now present a stylized example to illustrate these concepts. Consider the credit portfolio with four sectors given in Table 9. The first two sectors have a *PD* of 1% and exposure of 25; the other two sectors are lower *PD* (0.5%). For simplicity we assume a 100% *LGD.* The third and fourth column give the expected losses (*EL*) expressed in monetary terms and as percent of total *EL*. The following two columns give the computed stand-alone (SA) capital computed form the internal

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[19] This is consistent with Vasicek (2002), where it is shown that under the one-factor Merton model, one can approximate the losses of non-granular portfolios by applying the Vasicek formula using ($\rho + (1-\rho)\delta$) in place of the actual correlation ρ, where δ is the Herfindahl index on the sector exposures. We can also use this approximation further to get the implied asset correlation ρ for the sector.

multi-factor model (total and percent). The last column shows the implied intra-sector correlations, obtained by inverting the stand-alone capital formula (5).

Portfolio	EAD	PD	EL	EL %	SA Capital (One-Factor)	SA Capital % (One-Factor)	Implied Rho
P1	25	1.0%	0.25	33.3%	3.4	35.3%	**20.1%**
P2	25	1.0%	0.25	33.3%	2.1	21.5%	**12.4%**
P3	40	0.5%	0.20	26.7%	3.8	39.6%	**21.9%**
P4	10	0.5%	0.05	6.7%	0.4	3.7%	**8.6%**
Total	**100**		0.75	100.0%	**9.7**	100.0%	
					CDI	**32.9%**	

Table 9. Four-sector portfolio: characteristics and stand-alone capital

The portfolio total exposure is 100, the *EL* is 75bps and the stand-lone capital is 9.7%. The *CDI* is close to one third, implying that there are roughly three effective sectors. We can start understanding the effect of various credit parameters by comparing the contributions to total exposure, *EL* and SA capital. The differences in exposure and *EL* contributions can be explained by the interaction of the exposures with the *PDs* and *LGDs*. The intra-sector correlations explain the differences between *EL* and capital contributions. For example, the fourth sector represents one tenth of the exposures, almost 7% of *EL*, but less than 4% of the capital. This indicates that it is first a low *PD* sector and also that it has a lower than average implied intra-sector correlation. Consider, in contrast, the third sector portfolio, which constitutes 40% of the total exposure, 27% of *EL* and about 40% again of SA capital. This sector's low *PD* reduces its *EL* contribution, but its higher implied asset correlation (22%) increases its share of SA capital. The first sector's high capital contribution is explained by both high *PD* and intra-sector correlation.

Table 10 summarizes the results for overall economic capital and implied sector factor correlations. First, the multi-factor economic capital model is used to compute the overall economic capital, which is then used to calculate the *DF* and average sector factor implied correlation. The economic capital is 7.3% of the total exposure, implying a diversification factor *DF* = 75.5% (7.3 = 0.755 x 9.7). We use the tables from the previous section to estimate the average correlation $\bar{\beta}$; a correlation of 40% gives *DF=68%* and a correlation of 60% gives *DF=78.2%*. Using linear interpolation, we find the implied average correlation to be $\bar{\beta} = 54.9\%$.

Portfolio	Exposures	SA Capital (One-Factor)	Implied Rho	Capital % (Flat Beta=54.6%)	Economic Capital %	Implied Qk's
P1	25	35.3%	20.1%	36.1%	31.9%	**45.7%**
P2	25	21.5%	12.4%	19.0%	17.2%	**49.7%**
P3	40	39.6%	21.9%	42.3%	47.5%	**65.6%**
P4	10	3.7%	8.6%	2.6%	3.4%	**66.8%**
Total	100					

SA Capital	CDI	DF	Capital	Implied Average Beta
9.7	32.9%	**75.5%**	**7.3**	**54.9%**

Table 10. Multi-factor capital and implied correlations

The fifth column of Table 10 gives the capital contributions assuming that all sector factor correlations are equal to the average of 54.9%. These contributions are close but do not equal the SA capital contributions. In this case, every sector factor is equally correlated with the overall portfolio, and the only difference stems from the size component of the sector diversification factor ΔDF_k^{size} . The decomposition of the sector diversification factor for the case of a flat correlation is given on the left side of Table 11. Compared to the stand-alone case, the size component of the sector diversification factor increases contributions for the two biggest sectors (P1 and P3) and decreases them for the two small ones (P2 and P4). While the overall diversification factor is 75.6%, the marginal sector diversification factors range from 53% (P4) to 81% (P3).

	Flat Sector Factor Correlation (Average)				Implied Sector Factor Correlations			
Portfolio	DF_k	Portfolio Diversification	Sector Size	Sector Correlation	DF_k	Portfolio Diversification	Sector Size	Sector Correlation
P1	77.5%	75.6%	1.8%	0%	68.4%	75.6%	1.8%	-9.1%
P2	66.9%	75.6%	-8.7%	0%	60.7%	75.6%	-8.7%	-6.2%
P3	80.8%	75.6%	5.2%	0%	90.6%	75.6%	5.2%	9.8%
P4	53.3%	75.6%	-22.3%	0%	70.7%	75.6%	-22.3%	17.4%

Table 11. Decomposition of marginal sector diversification factors.

Next, the multi-factor economic capital model is used to compute the marginal capital contributions, and implied $\overline{Q}_k$'s for each sector are then estimated (see the last two columns of Table 10). For the first two sectors, the capital contributions are lower than those with equal correlations. Hence, we obtain lower than average implied correlations of the factors to the rest of the portfolio. The right half of Table 11 gives the decomposition of the sector diversification factors. Also, from the last column, we see that the first two sectors have negative sector correlation diversification components. The opposite is true for P3 and P4 (higher than average implied correlations and positive correlation component in the sector diversification factor).

The fitted *DF* model can now be used to calculate, almost instantaneously, sensitivities or the capital contribution of new loans or trades, while allowing us also to explain the sources of risk and diversification. For example, bringing in a new small exposure to sector 3, would result in a marginal capital contribution of about 90bps per unit of exposure (this is the product a marginal sector diversification of 90.6%, and a SA capital contribution of 39.6% divided by 40, or about 1% . The benefit of diversification is smaller given that the exposure is coming into a large, highly correlated sector, as explained earlier. Note that one can also use the model to compute the capital contributions of bigger transactions.

6. Concluding Remarks

We present a simple adjustment to the single-factor credit capital model, which recognizes the diversification obtained from a multi-factor credit setting. In contrast to full MC methods, there are benefits for seeking analytical or semi-analytical approximations for both for regulatory purposes as well as for credit portfolio decision management support tools. As a risk management tool, the model can be used to understand concentration risk, capital allocation and sensitivities, as well as to compute "real-time" marginal risk contributions for new deals or portfolios.

The model is based on the concept of a *diversification factor*. We estimate this diversification factor for a family of multi-factor models, and show that it can be expressed as a function of two parameters that broadly capture the size concentration and the average cross-sector correlation. The model further supports an intuitive capital allocation methodology. For this purpose we define *marginal diversification factors* at the obligor or sub-portfolio level, which account for their diversification contributions to the portfolio.

While, as presented, the estimation of the diversification factor requires substantial numerical work, it can then be tabulated and used readily as a basis for regulatory rules or economic capital allocation. This results in a practical, simple and fast, method that can be also applied for stress testing and pre-deal analytics. For example, an institution can re-calibrate the model using an advanced credit portfolio framework on a periodic basis (for example monthly, weekly and even daily) to adjust for changing market conditions and portfolio composition. The model can then be used in real time during the day to support decision making, origination and trading.

We believe the diversification factor has potential to be applied to extend the Basel II regulatory framework to a general multi-factor setting, thus allowing for more accurate model of diversification for portfolios across various asset classes, sectors and regions, and in particular within mixed portfolios in developed and emerging economies. However, a few remarks are appropriate with respect to its calibration together with the regulatory parameters from Basel II. While we have used in Section 4 the Basel formulae for wholesale exposures in these exercises, we do not wish to imply that, as presented, the calibration exercises are generally appropriate for regulatory rules. An explicit assumption of the results is that the underlying credit model is given by equations (2) and (3). The calibration of Basel II parameters was done generally in the context of a one-factor model. Thus, one can argue that, if the sample used for calibration already covers the sectors in the portfolio, the asset correlations ρ_k already account, at least partially, for cross-sector diversification (see also e.g. Lopez 2004). To the degree that the original parameter calibration accounts for cross sector diversification, some scaling (up) for intra-sector correlations or (down) the diversification factor is required, in order to not incur in double counting.

Finally, there are several enhancements of the model, which can be addressed in future research. These include:

- The *DF* presented only covers systemic credit risk (as does the Basel II model) and, hence, is most useful for large portfolios. Its current strength is on capturing sector and geographical, but not name (or counterparty), concentrations. A useful extension of the model would also cover idiosyncratic risk (name concentrations) by applying mathematical tools such as the granularity adjustment technique.
- There is potential for improving and generalizing the parameterization of the model. More parameters can be added or perhaps one can search for parameters that get better or more general fits (for example, the correlation parameter used in this paper is only one of several, which could have been chosen). However, in our opinion, this should not be done at the expense of too much complexity or loosing the intuitive interpretation of its parameters, results and capital allocation.
- The final model can also be potentially enhanced through a parameterization with an explicit functional dependence of the *DF* on the *CDI* and correlation.

- We have formulated how risk concentrations work within this type of model. Further work is needed to explore their mathematical behaviour, their role in model calibration and application in practice.
- Perhaps the biggest limitation of the model today is its reliance on costly numerical calibration. Ideally, we would like also a closed form approximation for the *DF* that is accurate and perhaps does not rely as much on numerical calibration. As such, for example, the known solution for Normal distributions can provide useful insights into the more general problem.

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Appendix. Correlations of Market Indices in Developed and Emerging Markets.

Matrix of correlations between stock market indices (7 years of monthly data)

	Spain	France	Germany	U.K.	Italy	USA	Canada	Japan	Argentina	Mexico	Brazil	Poland	Turkey	Malaysia	Thailand	Czech Rep.	Philippines
Spain	1.00	0.84	0.83	0.78	0.86	0.75	0.72	0.56	0.40	0.57	0.56	0.51	0.39	0.20	0.25	0.42	0.36
France	0.84	1.00	0.90	0.89	0.85	0.87	0.79	0.57	0.28	0.51	0.50	0.56	0.41	0.21	0.25	0.45	0.34
Germany	0.83	0.90	1.00	0.79	0.84	0.78	0.75	0.57	0.37	0.57	0.55	0.55	0.51	0.28	0.28	0.46	0.32
U.K.	0.78	0.89	0.79	1.00	0.76	0.89	0.74	0.48	0.26	0.51	0.46	0.52	0.39	0.20	0.33	0.34	0.39
Italy	0.86	0.85	0.84	0.76	1.00	0.70	0.65	0.59	0.36	0.52	0.55	0.52	0.48	0.14	0.28	0.45	0.30
USA	0.75	0.87	0.78	0.89	0.70	1.00	0.84	0.48	0.32	0.59	0.54	0.48	0.43	0.24	0.31	0.42	0.45
Canada	0.72	0.79	0.75	0.74	0.65	0.84	1.00	0.51	0.44	0.68	0.60	0.55	0.45	0.42	0.38	0.49	0.52
Japan	0.56	0.57	0.57	0.48	0.59	0.48	0.51	1.00	0.30	0.48	0.46	0.49	0.44	0.19	0.27	0.46	0.19
Argentina	0.40	0.28	0.37	0.26	0.36	0.32	0.44	0.30	1.00	0.62	0.54	0.37	0.30	0.39	0.38	0.34	0.42
Mexico	0.57	0.51	0.57	0.51	0.52	0.59	0.68	0.48	0.62	1.00	0.68	0.42	0.44	0.35	0.37	0.33	0.42
Brazil	0.56	0.50	0.55	0.46	0.55	0.54	0.60	0.46	0.54	0.68	1.00	0.45	0.48	0.30	0.33	0.36	0.47
Poland	0.51	0.56	0.55	0.52	0.52	0.48	0.55	0.49	0.37	0.42	0.45	1.00	0.32	0.42	0.36	0.62	0.41
Turkey	0.39	0.41	0.51	0.39	0.48	0.43	0.45	0.44	0.30	0.44	0.48	0.32	1.00	0.12	0.19	0.36	0.10
Malaysia	0.20	0.21	0.28	0.20	0.14	0.24	0.42	0.19	0.39	0.35	0.30	0.42	0.12	1.00	0.58	0.36	0.54
Thailand	0.25	0.25	0.28	0.33	0.28	0.31	0.38	0.27	0.38	0.37	0.33	0.36	0.19	0.58	1.00	0.22	0.55
Czech Rep.	0.42	0.45	0.46	0.34	0.45	0.42	0.49	0.46	0.34	0.33	0.36	0.62	0.36	0.36	0.22	1.00	0.33
Philippines	0.36	0.34	0.32	0.39	0.30	0.45	0.52	0.19	0.42	0.42	0.47	0.41	0.10	0.54	0.55	0.33	1.00

Each country's average correlation with the different economic groups (emerging/non-emerging)

	Spain	France	Germany	U.K.	Italy	USA	Canada	Japan	Argentina	Mexico	Brazil	Poland	Turkey	Malaysia	Thailand	Czech Rep.	Philippines
Average correlation with non-emerging economies	76%	82%	78%	76%	75%	76%	72%	54%	34%	55%	53%	52%	44%	23%	29%	44%	36%
Average correlation with emerging economies	41%	39%	43%	38%	40%	42%	50%	37%	42%	45%	45%	42%	29%	38%	37%	37%	40%