Corporate Bonds: Valuation, Hedging, and Optimal Call and Default Policies

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February 18, 2000

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Abstract

This paper studies the valuation and risk management of callable, defaultable bonds when both interest rates and firm value are stochastic and when the issuer follows optimal call and default policies. Since interest rate sensitivity is low when call is imminent and firm value sensitivity is high when default is imminent, characterizing the issuer's call and default policies is essential to understanding corporate bond risk management. We develop analytical results on optimal call and default rules and use them to explain the dynamics of a hedging strategy for corporate bonds using Treasury bonds and issuer equity.

To clarify the interaction between the issuer's embedded call and default options, we compare the callable defaultable bond to its pure callable and pure defaultable counterparts. Each bond's embedded option is a call on a riskless, noncallable host bond, distinguished only by its strike price. This generalized call option perspective generates intuition for a variety of results. For instance, spreads on all bonds, not just callables, narrow with interest rates; a decline in rates can trigger a default; a call provision can increase the duration of a risky bond; a call provision increases equity's sensitivity to firm value, mitigating the underinvestment problem identified by Myers (1977).

Corporate bonds are common components of balanced portfolios, yet the embedded options they contain are as difficult to value and hedge as many exotic derivatives. Even when the probability of bankruptcy is low, bond values are sensitive to changes in firm value, so corporate bond portfolio managers and investors always face the problem of managing interest rate and credit risk simultaneously. In addition, most corporate bonds are callable, as Duffee (1998) documents, and callable issues dominate benchmark bond indexes such as Moody's and Lehman Brothers'. The call provisions in these bonds interact with default risk in a complex fashion.

This paper examines the valuation and risk management of callable defaultable bonds when both interest rates and firm value are stochastic. We assume that the issuer follows rules for calling and defaulting that maximize the value of his equity. To our knowledge, this is the first model of corporate debt that incorporates both stochastic interest rates and endogenous bankruptcy. Although existing models of coupon-bearing corporate bonds either treat interest rates as constant or else impose exogenous default rules, these assumptions can be very restrictive. Bond yield spreads can be quite sensitive to interest rate volatility and correlation with firm value. Furthermore, the assumption of exogenous default rules can significantly narrow spreads.

The paper makes a number of contributions to the literature. First, we develop analytical results about the existence, shape, and relation of optimal call and default boundaries in the presence of stochastic interest rates. These results lead to new insights about the interaction between call provisions and default risk and provide the explanation for numerical results on hedging. Indeed, interest rate sensitivity is low when call or default is imminent. Firm value sensitivity is low when call is imminent and high when default is imminent. Therefore, characterizing the issuer's call and default policies is essential to understanding the risk management of corporate bonds.

Second, we conduct an extensive numerical study of the hedging of interest rate and credit risk using Treasury bonds and the issuer's stock. Our examples consider both high and low grade bonds. Although reduced-form models are flexible and tractable, structural models of corporate debt such as ours provide a unified framework for valuing debt and equity. We take advantage of this unified framework to generate intuition for the dynamics of the two-factor hedge.

Third, we establish a framework in which each of the three kinds of bonds, not just the pure callable bond and the pure defaultable bond, but also the callable defaultable bond, can be viewed as a noncallable, riskless host bond minus an American call option on that host bond. The strike price of the pure call option is the provisional call price, that of the pure default option is firm value, and that of the option to call or default is the minimum of the call price and firm value. This generalized call option perspective provides simple intuition for a variety of results. For example,

- spreads on all bonds, not just callables, narrow with interest rates

because all embedded option values decline with the value of the underlying host bond;

- credit spreads widen with the correlation between interest rates and firm value

because this increases the volatility of the payoff of the option to exchange the firm for the host bond. Longstaff and Schwartz (1995) have similar findings on spreads, but the mechanisms for the effects and their magnitudes are different. Shimko, et al. (1993) find the correlation effect in the case of a zero-coupon bond.

- A decline in interest rates can trigger a default

as the rise in the price of the underlying host bond puts the default option deeper in the money.

With regard to risk management, the call and default options have an interaction effect on duration. A call provision by itself reduces duration, as does default risk by itself, however,

- a call provision can increase the duration of a risky bond and
- default risk can increase the duration of a callable bond

because the presence of one option delays the exercise of the other.

- A call provision reduces a defaultable bond's sensitivity to firm value and increases equity's sensitivity to firm value,

because it makes the "strike price" of the embedded option less sensitive to firm value. The call provision therefore mitigates the underinvestment problem identified by Myers (1977).

Because of the complexity of the default and call options, much of the existing theory of defaultable debt treats interest rates as constant. Merton (1974) analyzes a risky zero-coupon bond and characterizes the optimal call policy for a callable coupon bond. Brennan and Schwartz (1977) model callable convertible debt. Black and Cox (1976), Geske (1977), Leland (1994), and Leland and Toft (1996) value coupon-paying debt when asset sales are restricted and solve for the equity holders' optimal default policy. Fischer, Heinkel, and Zechner (1989) and Leland (1998) also allow equity holders to restructure by calling the debt. Other models introduce costly liquidations and view bankruptcy as a bargaining process. These include Anderson and Sundaresan (1996), Huang (1997), Fan and Sundaresan (1997), Mella-Barral and Perraudin (1997), and Acharya, Huang, Subrahmanyam, and Sundaram (1999).

Models that introduce stochastic interest rates typically take a different approach to the treatment of bankruptcy. Some impose exogenous bankruptcy triggers in the form of critical asset values or payout levels. These include the models of Brennan and Schwartz (1980), Kim, Ramaswamy, and Sundaresan (1993), Neilsen, Saá-Requejo, and Santa-Clara (1993), and Longstaff and Schwartz (1995). Cooper and Mello (1991) and Abken (1993) model defaultable swaps assuming that equity holders can sell assets to make swap or bond payments. Other papers model default risk with a hazard rate or stochastic credit spread. See, for example, Ramaswamy and Sundaresan (1986), Jarrow, Lando, and Turnbull (1993), Madan and Unal (1993), Duffie and Singleton (1995), Jarrow and Turnbull (1995), Duffie and Huang (1996), and Das and Sundaram (1999).

Another literature analyzes callable bonds with stochastic interest rates in the absence of default risk. This includes Brennan and Schwartz (1977) and Courtadon (1982). Related work on American options on nondefaultable bonds includes Ho, Stapleton, and Subrahmanyam (1996), Jorgensen (1997), and Peterson, Stapleton and Subrahmanyam (1998). Amin and Jarrow (1992) provide a general analysis of American options on risky assets in the presence of stochastic interest rates.

This paper incorporates all of these elements: stochastic interest rates, stochastic firm value, call provisions, and optimal call and default rules. The instantaneous interest rate follows a one-factor Cox, Ingersoll, Ross (1985) process. The value of the assets of the firm is a diffusion with constant volatility. The firm has outstanding a callable, defaultable, fixed-rate bond of finite maturity. Asset sales are limited but equity holders can put up new money to service the debt. They can stop servicing the debt either by giving up the firm or else by calling the bond. The solution to the equity holder's optimal stopping problem determines the values of the equity and debt of the firm.

To clarify the interaction of the call and default options, we also model the callable defaultable bond's simpler counterparts: the noncallable defaultable bond with the same coupon, maturity and issuer, and the riskless callable bond with the same coupon, maturity and call provision. Each of the three bonds is a riskless noncallable host bond minus a generalized call option on the host bond. The three bonds differ only in the nature of strike price of their embedded options.

We establish a number of useful analytical results. Some of these are consistent with simple intuition. For example, the values of the bonds with embedded options are increasing in the price of the host bond and firm value. In addition, the value of levered equity is decreasing in the price of the host bond and increasing in firm value. Other results lead to new insights. For example, analytical results about the existence, shape, and relation of optimal exercise boundaries for the embedded options help explain numerical results on how hedge ratios change as market conditions change.

More specifically, we prove that at each point in time, t, prior to the bond maturity date, and for each level of firm value, v,

- there exists a critical host bond price $b(v, t)$ above which it is optimal to exercise the option.
- For the pure defaultable bond, the critical host bond price is increasing in v.
- For the callable defaultable bond, the critical host bond price is increasing in v in the default region, where v is less than the contractual call price k, and decreasing in v in the call region, where v is greater than k.

In addition, we describe the relationship between the different boundaries.

- For $v \leq k$, the critical host bond price for the callable defaultable lies above that for the pure defaultable bond. In other words, the default region is smaller for the callable defaultable than for the pure defaultable.
- For $v > k$, the critical host bond price for the callable defaultable lies above that for the pure callable bond. That is, the call region is smaller for the callable defaultable than for the pure defaultable.

With the combined call and default option, the value of preserving one option makes it optimal for the issuer to continue servicing the debt in states in which it would otherwise exercise the other option.

We then study the hedging of corporate bonds with Treasury bonds and issuer stock. The hedge ratios, or deltas, of the callable and defaultable bonds with respect to the host bond price are low when call or default is imminent. This is the case when either the host bond price is high or the critical exercise boundary $b(v, t)$ is low. The first effect means that increases in the host bond price reduce the deltas of the callable and defaultable bonds. The second effect means that the deltas of the callable and defaultable bonds, as functions of firm value, inherit the shape of the exercise boundaries described above. This explains a variety of properties of the risk measures. For example, the results on the relation between the different boundaries explain why a call provision can increase the duration of a defaultable bond at low firm values and why default risk can increase the duration of a callable bond at high firm values.

Next, we find that the presence of the call provision reduces the beta of a defaultable bond as well as its delta with respect to the issuer's equity. Nevertheless this delta grows without bound as firm value approaches zero. This highlights the difficulty of hedging the default risk of large positions in low grade bonds. Finally, we calculate equity deltas with respect to firm value to quantify the Myers (1977) underinvestment problem for firms with various capital structures.

The paper proceeds as follows. Section 1 describes the bonds with embedded options. Section 2 gives analytical results on valuation and numerical results on yield spreads. Section 3 gives analytical results on optimal call and default policies. Section 4 illustrates the effect of assuming constant interest rates or exogenous default rules. Section 5 studies corporate bond risk management. Section 6 quantifies the Myers underinvestment problem for the high grade and junk bond issuers in our examples. Section 7 concludes. Appendix 1 contains proofs and Appendix 2 describes the numerical method.

1 Bonds with Embedded Options

A riskless callable bond is equivalent to a riskless noncallable host bond minus an American call option on that host bond. The option gives the issuer the right to buy back the host bond at the contractual call price.

A defaultable bond can also be viewed as a host bond minus an option. In particular, in the absence of special covenants, equity holders control the firm as long as they service their debt. Even if asset cash flows are too small to meet coupon payments, equity holders may wish to raise new equity to service the debt if asset value is great enough (see, for example, Black and Cox (1976), Geske (1977), Leland (1994, 1998), and Leland and Toft (1996)). Thus, a shortage of asset cash flow need not trigger bankruptcy. The optimal default policy is endogenous. Default is essentially the option of equity holders to abandon the firm and its liabilities whenever they want, that is, the American option to buy back the promised debt cash flows in exchange for the firm. Consequently, in a simple capital structure with only one bond outstanding, a defaultable bond is equivalent to the host bond minus the American option to buy the host bond for the value of the firm.

If the defaultable bond is also callable, then it can be viewed as the host bond minus the equity holders' American option to buy the host bond for the minimum of the firm value and the contractual call price. These decompositions mean that once a particular term structure model is in place, analyzing a bond with an embedded option amounts to analyzing the option itself. This section describes these options formally.

Consider a complete, frictionless, arbitrage-free, continuous-time financial market with stochastic interest rates. Asset prices are stochastic processes defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with filtration $\{\mathcal{F}_t\}$. Let r_t represent the instantaneous spot interest rate at time t. Let $\tilde{\mathcal{P}}$ be the equivalent martingale measure under which the instantaneous expected rate of return on all assets at time t is r_t.

Assumption 1.1 *The interest rate r_t follows a continuous, adapted Cox-Ingersoll-Ross (1985) (CIR) process*

$$dr_t = (\alpha + \beta r_t)dt + \gamma\sqrt{r_t}d\tilde{Z}_t, \; t \in [0, \infty) \tag{1}$$

where $\tilde{Z}_t$ is a one-dimensional Brownian motion under $\tilde{\mathcal{P}}$, $r_0 > 0$, and $\alpha \geq 0, \beta$ and γ are real-valued constants.[1]

[1]In fact, the analytical results of this paper hold for any nonnegative interest rate process with coefficients that satisfy the Lipschitz conditions and for which the bond price function p defined in Lemma 1.1 is continuous in r.

Let β_t be the discount factor applicable for time t:

$$\beta_t \equiv e^{-\int_0^t r_s ds}. \tag{2}$$

Consider a firm with a single bond outstanding. The bond has a fixed continuous coupon c and maturity T. Without loss of generality, suppose the par value of the bond is one, and all other values are in multiples of this par value. We will consider alternately the case of a callable bond and that of a noncallable bond. The assets of the firm have value at time t given by V_t.

Assumption 1.2 *Firm value evolves according to the equation*

$$\frac{dV_t}{V_t} = (r_t - \delta(r_t, V_t, t))dt + \phi d\tilde{W}_t, \tag{3}$$

where $\tilde{W}_t$ is a one-dimensional Brownian motion under $\tilde{\mathcal{P}}$, with $d\langle \tilde{W}, \tilde{Z} \rangle_t = \rho dt$. The diffusion coefficient ϕ is constant. The proportional payout rate of the firm, δ, is non-negative and bounded.

The payout to the firm's equity holders at each moment in time is $\delta(r_t, V_t, t)V_t - c$. Situations in which this quantity is negative are those in which equity holders are raising new equity to service the debt in order to retain control of the firm. This ability of equity holders to put up new money is a standard feature of models with endogenous bankruptcy, such as Black and Cox (1976), Geske (1977), Leland (1994, 1998), and Leland and Toft (1996). We assume that bond covenants restrict payout and investment policies, so that equity holders cannot alter δ and ϕ.

Associated with the firm's bond is the noncallable, nondefaultable host bond with the same coupon, maturity, and par value. Let P_t denote the price of this host bond at time t. By definition of the equivalent martingale measure,

$$P_t = \tilde{E}[c \int_t^T e^{-\int_t^s r_u du} ds + 1 \cdot e^{-\int_t^T r_u du} \mid \mathcal{F}_t] \tag{4}$$

where $\tilde{E}[\cdot]$ denotes the expectation under the measure $\tilde{\mathcal{P}}$.

Lemma 1.1 *The process* $P_t = p(r_t, t)$ *for some bounded, continuous function* $p : \mathcal{R}^+ \times [0, T] \to \mathcal{R}$. *Furthermore,* $p(\cdot, t)$ *is strictly monotone and has a* C^1 *inverse.*

To develop an understanding of the callable defaultable bond, it is helpful to compare it to its simpler counterparts: the pure callable bond with the same coupon, maturity, and call price schedule, and the pure defaultable bond with the same coupon, maturity, and issuer. The three bonds differ only in the nature of their embedded options. We can use a single general framework to analyze all three options. Each embedded option is an American option to buy the noncallable, nondefaultable host bond for a strike price of K_t at any time prior to the expiration date T, the maturity date of the bond. The strike K_t has one of three possible specifications:

1. For the option embedded in the pure callable bond, $K_t = k_t$, a continuous, nonnegative, deterministic call schedule.

2. For the option embedded in the pure defaultable bond, $K_t = V_t$, the value of the firm.

3. For the option embedded in the callable defaultable bond, $K_t = k_t \wedge V_t$, the minimum of the call price k_t and the value of the firm V_t.

The *exercise value* of the option at time t, $0 \leq t \leq T$ is

$$g(P_t, V_t, t) \equiv (P_t - K_t)^+ \tag{5}$$

An *exercise policy* for the option is a stopping time of the filtration $\{\mathcal{F}_t\}$. At an arbitrary time t in the life of the option prior to the expiration date, the value of the option implied by adopting a given policy τ, $t \leq \tau \leq T$, is

$$J_t^\tau \equiv \frac{1}{\beta_t} \tilde{E}[\beta_\tau (P_\tau - K_\tau)^+ | \mathcal{F}_t]. \tag{6}$$

An *optimal* exercise policy maximizes the current option value. The optimal value of the option is

$$\zeta_t \equiv \sup_{t \leq \tau \leq T} \frac{1}{\beta_t} \tilde{E}[\beta_\tau (P_\tau - K_\tau)^+ | \mathcal{F}_t] \tag{7}$$

Under the interest rate and firm-value specifications, r and V are jointly Markov, the coefficients of r and $ln(V)$ satisfy Lipschitz conditions, and g is bounded. These properties, together with the invertibility of p in r, allow us to invoke Theorem 3.8 of Krylov (1980) to conclude that, given $P_t = p$, and $V_t = v$,

$$\zeta_t = f(p, v, t) \tag{8}$$

for some continuous function $f : \mathcal{R}^+ \times \mathcal{R}^+ \times [0, T] \to \mathcal{R}$, satisfying

$$f(p, v, t) \geq g(p, v, t). \tag{9}$$

The *optimal stopping time* τ is

$$\tau = \inf\{t \geq 0 : f(p, v, t) = g(p, v, t)\}. \tag{10}$$

The *continuation region* of the optimal stopping problem (7) is the open set

$$U \equiv \{(p, v, t) \in \mathcal{R}^+ \times \mathcal{R}^+ \times [0, T] : f(p, v, t) > g(p, v, t)\}. \tag{11}$$

At any point (p, v, t) in the continuation region U, it is optimal to continue rather than to exercise. The *exercise region* is $\overline{U}$, the complement of U in $\mathcal{R}^+ \times \mathcal{R}^+ \times [0, T]$.

2 Valuation

In this section we verify general properties of the values of the callable and defaultable bonds and their embedded options. We also compare the values of the three different embedded options and explore implications for yield spreads.

Assumption 2.1 *Suppose* $\exists \epsilon > 0$ *such that for* τ' *defined as*

$$\tau' \equiv \inf_{t \leq s \leq T}\{s : P_s \geq K_s + \epsilon\} \wedge T, \tag{12}$$

$\tilde{\mathcal{P}}(\tau' < T) > 0.$

Note that this holds for $K_t = V_t$ and $K_t = k_t \wedge V_t$. For $K_t = k_t$, we require k_t sufficiently small and $\sigma(r_t, t)$ non-trivial.

Theorem 2.1 *The following properties hold for all three embedded options.*

1. $f(p, v, t) > 0, \ \forall 0 \leq t < T.$
2. $p^{(1)} > p^{(2)} \Rightarrow f(p^{(1)}, v, t) > f(p^{(2)}, v, t).$
3. $v^{(1)} < v^{(2)} \Rightarrow f(p, v^{(1)}, t) \geq f(p, v^{(2)}, t).$
4. $p^{(1)} \neq p^{(2)} \Rightarrow \frac{f(p^{(2)},v,t)-f(p^{(1)},v,t)}{p^{(2)}-p^{(1)}} \leq 1.$ *(Call delta inequality)*
5. $v^{(1)} \neq v^{(2)} \Rightarrow \frac{f(p,v^{(2)},t)-f(p,v^{(1)},t)}{v^{(2)}-v^{(1)}} \geq -1.$ *(Put delta inequality)*

These results are consistent with basic intuition about option pricing. Each option is a call on the underlying host bond. The option value is strictly increasing in the value of this bond, but the rate of increase is bounded by one.

In the case of the pure defaultable bond, the embedded option can also be viewed as a put on assets of the firm. Thus, the option value is decreasing in the asset value, but the rate of change is bounded by minus one. In fact, the value of the pure default option is strictly decreasing in firm value.

The option embedded in the callable defaultable bond also inherits the put-like properties of the pure default option with respect to the firm value. Of course, the value of the option embedded in the pure callable bond does not depend on firm value.

Theorem 2.1 has direct implications for the values of the bonds with the embedded options and, in the case of the defaultable bonds, for the corresponding levered equity. Let $p_X(p, v, t)$ represent the value of the bond with the embedded option as a function of the host bond price, firm value, and time. The subscript X is C for the pure callable bond, D for the pure defaultable, and CD, for the callable defaultable. Then

$$p_X(p, v, t) = p - f_X(p, v, t) \ . \tag{13}$$

For the defaultable bonds, letting $e_X(p, v, t)$ represent the value of the levered equity as a function of the host bond price, firm value, and time,

$$e_X(p, v, t) = v - p + f_X(p, v, t) \ . \tag{14}$$

Note that maximizing the value of the option to call or default is equivalent to minimizing the value of the callable or defaultable bond and maximizing the value of the issuer's equity.

Corollary

1. *The prices of the bonds with embedded options, p_C, p_D and p_{CD}, are increasing in the host bond price p.*

2. *The prices of the defaultable bonds, p_D and p_{CD}, are increasing in firm value v.*

3. *The levered equity values, e_D and e_{CD}, are decreasing in the host bond price p.*

4. *The levered equity values, e_D and e_{CD}, are increasing in firm value v.*

The next result compares the values of the different options.

Proposition 2.1

$$f_C(p, v, t) \vee f_D(p, v, t) \leq f_{CD}(p, v, t) \leq f_C(p, v, t) + f_D(p, v, t).$$

The combined option to call or default is worth more than either of the simple options because it has a lower strike price. However, the combined option is worth less than the sum of simple options. This is because, with the combined option, calling destroys the default option, and defaulting destroys the call option. Kim, Ramaswamy, and Sundaresan (1993) call this the "interaction effect." In terms of yields to maturity, this means that the incremental spread created by a call provision will be less for a corporate bond then for a Treasury, as Kim, Ramaswamy, and Sundaresan (1993) observe. In addition, the interaction effect implies that the "option-adjusted" credit spread between a callable defaultable bond and its callable Treasury counterpart will be less than the credit spread of the noncallable issue.

2.1 Numerical Examples and Base Case Parametrization

Figures 1 and 2 illustrate the results of Theorem 2.1 and Proposition 2.1 with plots of option value, bond value and equity value as functions of the values of the underlying host bond and firm. Figure 2 also shows that as firm value grows large, the defaultable bond values converge to the values of their nondefaultable counterparts. To generate these and other numerical examples, we approximate the two-dimensional continuous-time model using a two-dimensional binomial lattice which extends the approach of Nelson and Ramaswamy (1990). Appendix 2 describes the numerical method in detail.

The plots on the left side of Figures 1 and 2 represent a "high grade" bond and the plots on the right represent a "junk" bond. Table 1 presents parameters and model values for the base case. The parameter values for the interest rate process are $\alpha =$ 0.034, $\beta =$ -0.5, and $\gamma = 0.10$, which are close to those used in Kim, Ramaswamy, and Sundaresan (1993). The initial interest rate is $r_0 = 5\%$. The base case firm payout rate, δ, is zero and firm volatility is $\phi = 0.20$. The base case correlation between interest rates and firm value, ρ, is zero.

Both the high grade and junk bonds have a maturity of 5 years and are currently callable at par. The high grade bond has a coupon rate of 6.25% and underlying firm value of 143% of bond par value. The junk bond has a coupon rate of 10.25% and underlying firm value of 118% of bond par value. This gives the high grade callable defaultable bond a yield spread of 145 basis points over the host bond and the junk bond a spread of 442 basis points. The high grade coupon is set low enough to ensure that its pure callable counterpart is still alive. The junk bond coupon is set high enough to give the incremental call provision in the defaultable substantial value, even though the default option is important. Consequently, the pure callable bond with the junk coupon is already called. The junk bond is more like a new issue, priced near par, while the high grade bond is more like a fallen angel, priced at a discount.

2.2 Yield Spreads

Practitioners typically quote corporate bond prices in terms of the spread of their yields over the yield of the comparable Treasury bond. Figure 3 plots the spreads of the yields to maturity of the callable and defaultable bonds over the host bond yield as a function of interest rates. Since the source of a bond's spread is its embedded option, the plots of spreads vs. interest rates in Figure 3 are like the mirror images of the plots of option values vs. host bond price at the top of Figure 1. As interest rates rise, the host bond price falls. Therefore, the embedded options, which are calls on the host bond, lose value, and the spreads narrow.

In the model of Longstaff and Schwartz (1995), spreads narrow with interest rate increases as well, but the mechanism is different. In their model, default occurs when firm value falls to an exogenous boundary, and in that event, bond holders receive a fraction of the value of the host bond. As interest rates rise, the drift of firm value under the risk-neutral measure increases, decreasing the probability of default, and this causes spreads to decline.

As Figure 3 shows, features of the relation between spreads and interest rates that our model generates are consistent with the patterns that Duffee (1998) finds empirically:

- spreads on all bonds, not just callables, narrow with rates;
- the relation is strongest for callables;
- among noncallables, the relation is stronger for lower grade bonds;
- for callables, the relation is stronger for higher priced bonds.

The results on the strength of the negative relation between spreads and interest rates can all be understood by viewing the embedded call or default option as a call on the host bond and recognizing that the option delta is highest when the option is deepest in the money. For these embedded options, that is the case when either the strike price is low, because a call provision is present or because firm value is low, or when the underlying host bond value is high.

3 Optimal Call and Default Policies

In this section we show that for each of the three types of bonds, pure callable, pure defaultable, and callable defaultable, a simple boundary separates the continuation region from the region of host bond and firm values in which it is optimal for the bond issuer to call or default. We also characterize the shape and relation of the different boundaries. The proofs of our results, in Appendix 1, are inspired by the analysis of the American put in Jacka (1991).

The first result establishes the existence of a boundary of critical host bond prices above which it is optimal for the issuer to call or default and below which it is optimal to continue servicing the debt. The result applies to all three types of bonds with embedded options.

Theorem 3.1 *For each $t \in [0, T)$ and $v > 0$, the (v, t)-section of U is*

$$U_{(v,t)} \equiv \{p : (p, v, t) \in U\} = (0, b(v, t)),$$

for some exercise boundary $b(v, t)$ satisfying $b(v, t) > K_t$. In some cases, $b(v, t)$ may exceed the upper bound on p.

In other words, for each time t and level of firm value v, if there is any bond price at which it is optimal to exercise the embedded option, then there is a critical bond price $b(v, t)$ below which it is optimal to continue and above which it is optimal to exercise the option. Figure 4 illustrates this critical host bond price boundary for each of the three bonds in both the high grade and junk bond cases. We use the labels b_C, b_D, and b_{CD} to distinguish the pure callable, the pure defaultable, and the callable defaultable. In this orientation, an increase in the host bond price, or a decline in interest rates, triggers the option exercise. While it is natural to think of interest rate declines triggering bond calls, Theorem 3.1 implies that interest rate declines can also trigger bond defaults.

Models with constant interest rates describe the optimal call and default rules in terms of critical firm values, an upper critical firm value above which call is optimal and a lower critical firm value below which default is optimal (see, for example, Merton (1974), Black and Cox (1976), and Leland (1994, 1998)). Our next result states that this characterization is also valid when interest rates are stochastic, only now the critical firm values are functions of interest rates.

Theorem 3.2 *For each $t \in [0, T)$ and $p > 0$,*

1. *the (p, t)-section of the continuation region for the pure default option, U_D, is*

$$U_{D(p,t)} \equiv \{v : (p, v, t) \in U_D\} = (v_D(p, t), \infty),$$

for some exercise boundary $v_D(p, t)$ satisfying $v_D(p, t) < p$;

2. *the (p, t)-section of the continuation region for the option to call or default, U_{CD}, is*

$$U_{CD(p,t)} \equiv \{v : (p, v, t) \in U_{CD}\} = (v^D_{CD}(p, t), v^C_{CD}(p, t)),$$

where $v^D_{CD}(p, t) \leq k_t \leq v^C_{CD}(p, t)$, and $v^D_{CD}(p, t) < p$. In some cases, $v^C_{CD}(p, t)$ may be infinite.

The next theorem describes the shape and relation of the different boundaries. Figure 4 illustrates these results.

Theorem 3.3 *For each $t \in [0, T)$,*

1. *$b_C(v, t)$ is constant in v.*

2. *$v^{(1)} < v^{(2)} \Rightarrow b_D(v^{(1)}, t) \leq b_D(v^{(2)}, t)$.*

3. *$p_1 < p_2 \Rightarrow v_D(p_1, t) \leq v_D(p_2, t)$.*

4. $v^{(1)} < v^{(2)} \leq k_t \Rightarrow b_{CD}(v^{(1)}, t) \leq b_{CD}(v^{(2)}, t)$.

5. $v \leq k_t \Rightarrow b_{CD}(v, t) > b_D(v, t)$.

6. $k_t < v^{(1)} < v^{(2)} \Rightarrow b_{CD}(v^{(1)}, t) \geq b_{CD}(v^{(2)}, t)$.

7. $v > k_t \Rightarrow b_{CD}(v, t) > b_C(v, t)$.

Part 1 of Theorem 3.3 just notes that the critical host bond price above which pure callable bond should be called does not depend on firm value. Now consider the default option embedded in the pure defaultable bond. Part 2 states that the critical bond price above which the firm should default, $b_D(v, t)$, is increasing in the firm value v. That is, the higher the firm value, the lower the interest rates must be to trigger a default. Conversely, Part 3 indicates that the critical firm value below which the equity holders should default is increasing in the host bond price p. In other words, in high interest rate environments, it takes lower firm values to make equity holders stop servicing the debt and give up control of the firm.

Finally, consider the option to call or default that is embedded in a callable defaultable bond. Fix a date t. For firm value below the call price, $v < k_t$, exercising the option means defaulting. For firm value greater than the call price, $v > k_t$, exercising means calling the bond. First, consider the region $v \leq k_t$. Part 4 of Theorem 3.3 indicates that the critical host bond price, $b_{CD}(v, t)$, above which it is optimal to default, is increasing in v, like $b_D(v, t)$. Part 5 states that the callable defaultable has a larger continuation region than the noncallable defaultable. In other words, the issuers of a callable defaultable will service the bond longer than they would if the bond were noncallable, because defaulting means throwing away the value of the call.

Now consider the region of high firm values, $v > k_t$, where exercising means calling the bond. Part 6 of Theorem 3.3 indicates that the critical host bond price, $b_{CD}(v, t)$, above which it is optimal to call, is decreasing in v. That is, at lower firm values, it takes lower interest rates to trigger a bond call. Part 7 indicates that the callable defaultable will continue to be serviced at lower interest levels than the pure callable, because calling means throwing away the default option. These results will be useful for understanding patterns in the risk measures presented in section 5.

4 Alternative Default Rules and Constant Interest Rates

To our knowledge, this is the first model of corporate debt to incorporate both optimal default rules and stochastic interest rates. To illustrate the importance of these features, this section shows how alternative default rules or constant interest rates can affect yield spreads.

4.1 Alternative Default Rules

Structural models of corporate debt that introduce stochastic interest rates often assume that default occurs when firm value falls below an exogenously specified level, V^*. For example, in Brennan and Schwartz (1980), default occurs when firm value falls to the sum of the par value of the straight debt and a fraction of the par value of the convertible debt. In Kim, Ramaswamy, and Sundaresan (1993), the firm defaults when asset cash flow is insufficient to cover coupon payments. In Longstaff and Schwartz (1995), default occurs when firm value hits an exogenously specified level as well. Such constraints are appropriate if minimum net worth or cash flow covenants are present, but may distort bond values otherwise.

Table 2 shows bond yield spreads under various default policies. The first row contains the spreads in the base case with the optimal default rule, that which maximizes equity value. The second row contains spreads for the case in which the firm is forced to default when firm value falls to bond par value. At this firm value, equity would still have positive value under the optimal default policy, and equity holders would prefer to put up new money to retain control of the firm assets. For reference, at the base case level of interest rates and with five years to maturity, issuers of a callable defaultable bond would optimally default at $V = 66$ (percent of bond par value) in the high grade case and $V = 82$ in the junk case. The minimum net worth constraint can substantially reduce equity value and increase bond value. The table shows spreads narrowing 40 to 100 basis points.

The third row of Table 2 shows the effect of assuming an exogenous default boundary that is too low. The critical firm value below which default occurs is set to $V^* = 30$ for the high grade bond and $V^* = 40$ for the junk bond. In this case, equity values will be negative in states in which the model assumes they are servicing the debt although they would rather abandon the firm. Again, as the examples show, this can substantially narrow spreads. In fact, the callable junk bond becomes so valuable under this policy that it is optimal for the issuer to call it.

The fifth row of Table 2 shows the effect of a cash flow constraint. The firm defaults when asset cash flow δV falls below the coupon c, that is, when V falls below c/δ. For this example, we introduce a positive asset payout rate, $\delta = 7.8\%$ for the high grade case, and $\delta = 10.8\%$ for the junk case, giving default boundaries of $V^* = 80$ and $V^* = 95$, respectively. For comparison, the fourth row of Table 2 shows spreads when equity holders can follow the optimal default rules given these same payout rates. Again, the minimum net cash flow constraint forces equity holders to relinquish the firm when they would otherwise put up new money and can dramatically increase bond value. In the examples, the cash flow constraint narrows spreads 40 to 150 basis points.

4.2 Constant Interest Rates

Existing models of corporate debt with endogenous default policies, such as Black and Cox (1976) and Leland (1994), take interest rates to be constant. In the examples of Brennan and Schwartz (1980) and Kim, Ramaswamy, and Sundaresan (1993), the assumption of constant interest rates has only a slight impact on spreads. However, such a result cannot be generalized. In our examples, introducing stochastic interest rates can materially affect pricing.

The bottom panel of Table 2 shows spreads for the case in which interest rates are constant at the host bond yield and for three cases in which interest rates are stochastic. In all cases, the firm follows optimal call and default policies. The impact of stochastic interest rates on spreads depends on the correlation between firm value and interest rates, ρ. We consider the cases $\rho = -0.50$, 0, and 0.50.

The examples suggest that with zero correlation, interest rate volatility widens spreads. Furthermore, spreads increase with ρ. The intuition for these effects comes from considering the source of the spread, the embedded option. The embedded option payoff is $(P - V)^+$ or $(P - V \wedge k)^+$, where P is the host bond price. The value of the option increases with the volatility of the difference $P - V$ or $P - V \wedge k$. With zero correlation, introducing volatility in P increases the volatility of the difference. Introducing positive correlation between firm value and interest rates leads to negative correlation between P and V and increases the volatility of their difference. This makes the option more valuable and bond spreads wider.

Shimko, et al. (1993) and Longstaff and Schwartz (1995) also find that spreads widen with the correlation between firm value and interest rates. Shimko, et al. (1991) analyze a zero-coupon bond. Longstaff and Schwartz (1995) model coupon-bearing bonds, as we do, but the mechanism for their correlation effect is different. When innovations in firm value correlate positively with interest rates, then they correlate positively with the drift of firm value under the risk-neutral measure, which increases the total variance of firm value. This increases the probability that firm value hits the default boundary and widens spreads. In the event of default, bond holders in the Longstaff and Schwartz model get a fraction of host bond value, whereas in our model, they get the value of the firm. A comparison of their examples with ours suggests that the magnitude of the correlation effect is much greater in our model, where bond payoffs in default depend on firm value.

5 Hedging Interest Rate Risk and Credit Risk

Defaultable bonds are subject to both interest rate risk and credit risk. In principle, a portfolio containing Treasury bonds and shares of the bond issuer's equity could serve to hedge both risks. By modeling a firm's debt and equity values in a single, internally consistent framework, we can gain intuition about the dynamics of this two-factor hedge.

Although the hedge ratios explicitly spell out the strategy for hedging a corporate bond, practitioners typically use the concepts of duration to measure interest rate risk and beta to measure stock market risk. This section therefore studies both host bond and equity hedge ratios and the more traditional risk measures duration and beta. Not surprisingly, the traditional risk measures share many of the properties of the hedge ratios. Thus, intuition about the hedging from option theory yields new insights about duration and beta.

5.1 Definition of Hedge Ratios

We define the hedge ratio or *delta* of a given asset with respect to a given hedging instrument as the number of units of the instrument to combine with a short position in the asset to make the return on the resulting hedged position uncorrelated with the hedging instrument. In the two-dimensional binomial lattice described in Appendix 2, every asset or portfolio has one of four possible values at the end of the first period. We measure the delta of a given asset with respect to a given hedging instrument as the slope coefficient in a regression of the future asset values on the future hedging instrument values. We consider two hedging instruments, the host bond and the issuer's equity. We also consider one-factor hedges using only the host bond or only the equity as well as two-factor hedges using the host bond and equity simultaneously. The one-factor deltas correspond to the simple regression betas and the two-factor deltas correspond to the multiple regression betas.

Table 1 contains base case values for the one- and two-factor deltas of the callable and defaultable bonds with respect to the host bond and the equity, while Figures 5 and 6 plot these deltas as functions of the host bond price and firm value. In the figures, the two-factor deltas are labeled "Mdel" while the one-factor deltas are labeled "Sdel." Even though the examples set the instantaneous correlation between interest rates and firm value, ρ, equal to zero, the value of the levered equity is negatively correlated with the host bond price. Indeed, the value of the levered equity is a decreasing function of the host bond price, as stated in part 3 of the corollary to Theorem 2.1. Therefore, as the figures show, the two-factor deltas with respect to the host bond must be greater than the one-factor deltas, in order to compensate for the short host bond position in the equity. Similarly, the two-factor deltas with respect to the equity exceed the one-factor deltas. For example, a position using only host bonds and no equity to hedge the noncallable junk bond would short 54 par amount of host bonds for each 100 par of junk bonds, but a position using both host bonds and equity would short 81 par amount of host bonds.

5.2 Deltas with Respect to the Host Bond

Figure 5 plots the deltas with respect to the host bond as a function of both the host bond price and the firm value. The dynamics of the callable and defaultable bond deltas are driven by the deltas of their embedded options. Each option is a kind of call on the host bond. The delta of the call is high when the exercise of the option is imminent. This is the case when either the host bond price is high or when the critical host bond price boundary $b(v,t)$, above which call or default occurs, is low. The first effect, that of a high host bond price, means that increases in host bond price increase the option deltas. Consequently,

- the deltas of the callable and defaultable bonds are decreasing in the host bond price.

This is visible in the plots of deltas vs. the host bond price in the top of Figure 5.

The second effect, that of a low exercise boundary, has more complex implications. The lower the boundary, the more imminent an option exercise, the higher the option delta, the lower the delta of the callable or defaultable bond. Therefore, the deltas of the callable and defaultable bonds, as functions of firm value, essentially inherit the shape of their exercise boundaries, $b(\cdot, t)$, described in Theorem 3.3 and illustrated in Figure 4:

- The delta of the pure defaultable bond is increasing in firm value.
- The delta of the callable defaultable bond is increasing in firm value for low firm values and decreasing in firm value for high firm values.
- The delta of the callable defaultable bond exceeds that of the pure defaultable for low firm values.
- The delta of the callable defaultable bond exceeds that of the pure callable for high firm values.
- The deltas of the defaultable bonds go to zero as firm value goes to zero.
- The delta of the callable defaultable bond converges to that of the pure callable as firm value grows large.

These effects are apparent in the plots of deltas vs. firm value in the middle of Figure 5.

The interaction of the call and default options has a surprising effect on the delta with respect to the host bond. First, whereas the call provision uniformly reduces the delta of a riskless bond, that is, the delta of the pure callable is uniformly less than one, the call provision can actually increase the delta of a defaultable bond. The middle right plot of Figure 5 shows that at low firm values the delta of the callable defaultable bond exceeds that of the pure defaultable. This is where the defaultable bond is near default but the callable defaultable bond continues to be serviced because the issuer want to preserve the call option, as Theorem 3.3, part 5 describes.

Second, whereas default risk uniformly reduces the delta of a noncallable bond, that is, the pure defaultable bond delta is always less than one, default risk can substantially increase the delta of the callable bond at high host bond values and high firm values. Indeed, the top plots of Figure 5 and the middle left plot all show regions where the delta of the callable defaultable bond exceeds that of the pure callable. This happens because at high host bond values and moderately high firm values, the pure callable gets called, but the the issuer of the callable defaultable continues debt service to preserve the default option, as Theorem 3.3, part 7 describes. The next section shows that these interaction effects exist for durations as well.

Duration

For noncallable, nondefaultable bonds, duration, or more specifically, modified duration, approximates the percent change in price given a 100 basis point change in yield:

$$dur \equiv -(\frac{dP}{dY})/P \ . \tag{15}$$

To extend this definition to the bonds with embedded options, we compute duration numerically as

$$Dur \equiv -(\frac{P_X(r+\varepsilon) - P_X(r)}{Y_H(r+\varepsilon) - Y_H(r)})/P_X \ , \tag{16}$$

where Y_H is the yield on the host bond, and the subscript X can be H, C, D, or CD, corresponding to the host bond, the pure callable, the pure defaultable, or the callable defaultable.

The sixth row of Table 1 contains durations of the different bonds. The duration of the five-year 6.25% host bond is about 4.3 in the base case. The callable or high grade defaultable bonds with the same coupon and maturity have durations in the range of 3.4 to 3.6. The duration of the 10.25% host bond is 4.0 while the junk bonds with the same coupon and maturity are significantly lower: the pure defaultable has duration of 2.5 and the callable defaultable has duration of 1.3.

The bottom left plot of Figure 5 shows how the durations of the high grade bonds vary with the instantaneous interest rate r. The host bond duration decreases with interest rates, reflecting its slight positive convexity. By contrast, the bonds with embedded options display negative convexity; their durations can increase with the interest rate, quite sharply in the callable cases.

The plots at the bottom of Figure 5 show that the interaction of the call and default options affects durations, as well as spreads and deltas. Alone, the pure default option and the pure call option each reduce duration. But in the presence of one option, the incremental effect of the other option can be to increase duration. For example, the bottom left plot of duration vs. interest rates, which is like the mirror image of the top left plot of delta vs. host bond price, shows that although default risk uniformly reduces the duration of the noncallable bond, it significantly increases the duration of the callable bond when rates are low. This is because the presence of the default option makes the issuer wait longer to call the bond.

The bottom right plot of Figure 5 shows how the durations of the junk bonds vary with firm value. The graphs of the defaultable bond durations are similar in shape to those of the deltas in the middle right plot. The pure defaultable bond duration increases with firm value as default becomes improbable. The duration of the callable defaultable goes to zero both as firm value falls to the default trigger and as it rises to the call trigger. Yet, again, at low firm values, the call provision actually increases the duration of the defaultable bond, because it makes the issuer continue longer.

5.3 Deltas with Respect to Issuer Equity

Figure 6 plots the deltas of the defaultable bonds with respect to the value of their issuers' equity. With regard to scale, the delta represents the fraction of the entire equity of the firm necessary to short in order to hedge a long position in the entire defaultable debt of the firm. A number of effects are apparent in Figure 6. First,

- the callable defaultable bond has a lower delta than the pure defaultable.

The incremental call provision in the callable defaultable bond reduces the bond's delta because it dampens the sensitivity of the embedded option strike price with respect to firm value (the option payoff is $(P - V \wedge K)^+$ instead of $(P - V)^+$).

Second, as default becomes imminent, the sensitivity of the option values to firm value increases in magnitude. This has two implications. One, illustrated in the plots of deltas vs. host bond value in the top of Figure 6, is that

- as the host bond price increases, the delta of the pure defaultable bond increases.

The other, visible in the plots of deltas vs. firm value in the middle of Figure 6, is that

- as firm value declines to zero, the deltas of both the defaultable bonds increase to infinity.

This raises issues about the equilibrium impact and feasibility of hedging default risk. Hedging a long position in defaultable bonds necessitates short positions in both the host bond and the issuer's equity. Taking those short positions increases the total supply of the host bond and equity, and therefore alters equilibrium prices. One might expect the equilibrium effect of shorting a Treasury bond to be small. However, the large short positions in equity necessary to hedge defaultable bonds with nontrivial default risk could have an important price impact. Therefore, while our deltas with respect to equity do illustrate the degree of equity exposure a defaultable bond imparts, for the practical purpose of hedging default risk, our analysis is only valid for small positions in defaultable bonds. For large positions, our analysis merely quantifies the difficulty of hedging default risk.

Beta

To see how much stock market risk the defaultable bonds and their corresponding levered equity inherit from the firm value, this section examines the betas of these instruments under the assumption that the beta of the firm value is one. In the base case, in which the correlation between interest rates and firm value is zero, the instantaneous beta of each of these instruments is, by Itô's lemma, just the elasticity of the value of the instrument with respect to firm value.

Table 1 lists the betas of stock and defaultable bonds in the base cases. The high grade bond betas are 0.16 for the pure defaultable and 0.14 for the callable defaultable, while the junk bond betas are higher, 0.38 for the pure defaultable and 0.21 for the callable defaultable. The stock of the more highly levered issuer also has a much higher beta. The high grade issuers have equity betas of about 2.7, whereas the junk issuers have equity betas about twice as high: 5.9 for the pure defaultable and 5.1 for the callable defaultable.

The bottom plots in Figure 6 show how the junk bond betas vary with interest rates and firm value. The beta measure behaves much like the delta with respect to equity. The bottom left plot of bond betas vs. the interest rate is like the mirror image of the top right plot of deltas vs. the host bond price. The bottom right plot of beta vs. firm value is like the middle right plot of delta vs. firm value. The presence of the call provision appears to reduce the bond beta uniformly, as it does the delta of the bond with respect to the equity value.

6 Leverage and Underinvestment

Myers (1977) demonstrates that the presence of risky debt can lead equity holders to pass up positive net present value projects and thus fail to maximize the value of the firm. In this section we use our model to quantify the magnitude of the underinvestment problem for the issuers of the high grade and junk bonds in our examples.

In the problem of choosing among discretionary investments, the policy that maximizes firm value V is to invest whenever the incremental increase in firm value V exceeds the increment in the level of investment I. That is, invest as long as $dV/dI > 1$. However, if equity holders must put up all the funds for the investment, but share the increase in firm value with the debt holders, then, as Myers (1977) argues, their choice will be to invest whenever the increment in equity value, V_E, exceeds the incremental investment. That is, levered equity holders will invest as long as $dV_E/dI > 1$ or, in terms of the delta of equity with respect to firm value, as long as $dV/dI > 1/(\partial V_E/\partial V)$. Since the delta of equity with respect to firm value, $\partial V_E/\partial V$, is less than one when debt is risky, the required profit margin for discretionary projects is greater than zero, leading to underinvestment and suboptimal firm value.

We can use our model to gauge the magnitude of the underinvestment problem for issuers of different credit quality. Table 1 contains equity deltas with respect to firm value for the issuers of the high grade and junk quality defaultable and callable defaultable bonds. For the high grade issues, which have a yield spread of 120 to 145 basis points, the corresponding equity deltas are around 0.9, suggesting that projects have to be worth at least 10% more than they cost in order to be attractive to equity holders.

The problem is much more severe for the junk issues. The pure defaultable junk bond, which is priced at 291 basis points over the corresponding Treasury bond, has an equity delta of 0.67. This suggests that the equity holders would require a profit margin of at least 50% on discretionary investments, potentially passing up projects with substantial net present value. The incremental call provision of the callable defaultable bond increases the yield spread to 442 basis points, but it reduces the sensitivity of risky debt value to changes in firm value, as Section 5 explains. Thus, the call provision increases the equity delta to 0.83, so it reduces the required profit margin to 20%, mitigating the underinvestment problem.

To see why the call provision increases the equity delta with respect to firm value and thus mitigates the underinvestment problem, recall that the equity values for noncallable and callable debt, are, respectively,

$$e_D(p, v, t) = v - p + \sup_{t \le \tau \le T} \frac{1}{\beta_t} \tilde{E}[\beta_\tau (P_\tau - V_\tau)^+ | \mathcal{F}_t] \tag{17}$$

and

$$e_{CD}(p,v,t) = v - p + \sup_{t \le \tau \le T} \frac{1}{\beta_t} \tilde{E}[\beta_\tau (P_\tau - V_\tau \wedge k_\tau)^+ | \mathcal{F}_t] \ . \tag{18}$$

Because of its modified "strike price," $V \wedge k$ instead of just V, the value of the option to call or default is less negatively related to firm value than the value of the pure call option. This makes the corresponding equity value more positively related.

The examples suggest that the impact of the call provision in reducing the underinvestment problem may be greater for lower grade bonds. This would have implications for differences in security design across issuers of different credit quality and warrants further study.

7 Summary and conclusion

Even if they do not deal with derivatives explicitly, corporate bond portfolio managers and investors encounter complex embedded options every day. This paper studies the valuation and risk management of callable, defaultable bonds when both interest rates and firm value are stochastic and when the issuer follows policies that maximize the value of his equity. Since the sensitivity of a bond to interest rates and firm value depends on how close the bond is to call or default, characterizing the issuer's call and default policy is crucial to understanding corporate bond risk management. We advance the theory of corporate bond valuation and optimal call and default rules and use it to explain the dynamics of a hedging strategy for corporate bonds using Treasury bonds and issuer equity.

Call provisions and default risk interact in subtle ways. To disentangle their effects, we compare the callable defaultable bond to its pure callable and pure defaultable counterparts. We develop a framework in which each bond's embedded option can be viewed as a call on a riskless, noncallable host bond, distinguished only by the nature of its strike price. This generalized call option perspective provides intuition for a variety of results on pricing, hedging, and optimal call and default policies.

We have focused on how changes in market conditions, that is, interest rates and firm value, affect prices, spreads, durations, betas, hedge ratios, and call and default decisions. However, many other issues surround the subject of corporate debt. One area of interest is term structure: how do changes in bond maturity affect call and default rules, spreads between Treasury and corporate par rates, and interest rate and equity sensitivity? Another concerns issues in corporate finance and security design: how do call provisions affect conflicts of interest between debt and equity holders and how do these effects vary across firms with different capital structures and asset risk profiles? The framework we have developed here could provide the foundation for research in a variety of different directions.

Appendix 1: Proofs

Proof of Lemma 1.1 By the Markov property of r, the conditional expectation in Equation (4) depends only on the level of the time t interest rate and time. The non-negativity of r ensures that p is bounded. Finally,

$$p(r_t, t) = c \int_t^T q(r_t, t, s) ds + q(r_t, t, T), \tag{19}$$

where $q(r_t, t, s) \equiv \tilde{E}[e^{-\int_t^s r_u du} \mid \mathcal{F}_t]$, is the time t price of the zero-coupon bond maturing at time s, given the time t interest rate r_t. Under the CIR specification of interest rates, the function $q(\cdot, t, s)$ is continuously differentiable and strictly monotone, and therefore, so is $p(\cdot, t)$. It follows that $p(\cdot, t)$ has a C^1 inverse. $\square$

The proof of Theorem 2.1 makes use of a number of so-called *no-crossing properties.* The first follows from Proposition 2.18 of Karatzas and Shreve (1987):

Proposition 7.1 *Consider two values of interest rates at time 0, $r_0^{(1)}$ and $r_0^{(2)}$ such that $r_0^{(1)} \leq r_0^{(2)}$, and denote the corresponding interest rate processes as $r_t^{(1)}$ and $r_t^{(2)}$, respectively. Then*

$$\tilde{\mathcal{P}}[r_t^{(1)} \leq r_t^{(2)},\ 0 \leq t < \infty] = 1. \tag{20}$$

This no-crossing property of r implies no-crossing properties for β, P, βP, and V:

Corollary 7.1 *Let $\beta_t^{(1)}$ and $\beta_t^{(2)}$ be the discount factor processes corresponding to initial interest rates $r_0^{(1)}$ and $r_0^{(2)}$, respectively. Then*

$$r_0^{(1)} < r_0^{(2)} \Rightarrow \beta_t^{(1)} > \beta_t^{(2)},\ \tilde{\mathcal{P}} - a.s.\ \forall\ 0 < t < \infty. \tag{21}$$

Proof From Proposition 7.1, we have $r_s^{(1)} \leq r_s^{(2)},\ \forall\ 0 \leq s \leq t$. The paths of $r^{(1)}$ and $r^{(2)}$ are continuous, so there exists a neighborhood around $t = 0$ on which $r^{(1)} < r^{(2)}$. Consequently, $e^{-\int_0^t r_s^{(1)} ds} > e^{-\int_0^t r_s^{(2)} ds}$. $\square$

The monotonicity of the discount factor in level of the interest rate implies:

Corollary 7.2 $r_0^{(1)} \leq r_0^{(2)} \Rightarrow P_t^{(1)} \geq P_t^{(2)},\ \tilde{\mathcal{P}} - a.s.\ \forall\ 0 \leq t \leq T.$

Combining Corollaries 7.1 and 7.2 yields:

Corollary 7.3 $r_0^{(1)} < r_0^{(2)} \Rightarrow \beta_t^{(1)} P_t^{(1)} > \beta_t^{(2)} P_t^{(2)},\ \tilde{\mathcal{P}} - a.s.\ \forall\ 0 \leq t \leq T.$

Under Assumption 1.2,

$$V_t = V_0 \cdot e^{\int_0^t r_u du - \int_0^t \delta_u du - \frac{\phi^2}{2} t + \phi \tilde{W}_t}. \tag{22}$$

It follows that:

Corollary 7.4 $r_0^{(1)} < r_0^{(2)} \Rightarrow V_t^{(1)} < V_t^{(2)},\ \tilde{\mathcal{P}} - a.s.\ \forall\ 0 \leq t \leq T.$

For use later, we introduce two other process, the *cum-coupon* bond price P^*, defined by the following equation:

$$\beta_t P_t^* \equiv \tilde{E}[c \int_0^T \beta_s ds + 1 \cdot \beta_T | \mathcal{F}_t],\ \forall\ 0 \leq t \leq T, \tag{23}$$

and the *cum-payout* value of the firm defined by

$$V_t^* \equiv V_t \cdot e^{\int_0^t \delta_u du}\ . \tag{24}$$

Both the processes βP^* and βV^* are martingales under the measure $\tilde{\mathcal{P}}$. Rewriting Equation (4) as

$$\beta_t P_t = \tilde{E}[c \int_t^T \beta_s ds + 1 \cdot \beta_T | \mathcal{F}_t], \tag{25}$$

we have

$$\beta_t P_t^* = \beta_t P_t + c\int_0^t \beta_s ds. \tag{26}$$

The following lemma also serves in the proof of Theorem 2.1.

Lemma 7.1 $r_0^{(1)} \le r_0^{(2)} \Rightarrow \tilde{E}[\beta_t^{(2)} P_t^{(2)} - \beta_t^{(1)} P_t^{(1)}] \ge P_0^{(2)} - P_0^{(1)},\ \forall 0 \le t \le T.$

Proof

$$\begin{aligned} \beta_t P_t - P_0 &= \beta_t P_t^* - c\int_0^t \beta_t dt - P_0 & (27)\\ \Rightarrow \tilde{E}[\beta_t P_t] - P_0 &= -\tilde{E}[c\int_0^t \beta_s ds]. & (28)\end{aligned}$$

Corollary 7.1 implies that

$$\tilde{E}[c\int_0^t \beta_s^{(1)} ds] \ge \tilde{E}[c\int_0^t \beta_s^{(2)} ds], \tag{29}$$

and the result follows. □

Finally, for ease of notation, we denote $\frac{\beta_\tau}{\beta_t}$ as $\beta_{t,\tau}$.

Proof of Theorem 2.1

1. The policy τ', defined in equation (12), is feasible for the optimal stopping problem (7). Therefore,

$$f(p, v, t) \ge \epsilon \cdot \tilde{E}[\beta_{t,\tau'} 1_{(\tau' < T)} | \mathcal{F}_t] > 0,$$

where $1_{(A)}$ is the indicator function for the set of events A.

2. Consider the stopping problem at time $t < T$. Let $p^{(1)} > p^{(2)}$ be two possible values of the time t bond price. Note that, from the strict monotonicity of $p(\cdot, t)$, there are corresponding values of the time t interest rate process, $r^{(1)}$ and $r^{(2)}$, satisfying $r^{(1)} < r^{(2)}$. Let τ be the optimal stopping time given the state at time t is $P_t = p^{(2)}$ and $V_t = v$. Then its feasibility as a stopping time for the state $P_t = p^{(1)}$ and $V_t = v$ implies that

$$f(p^{(1)}, v, t) - f(p^{(2)}, v, t) \ge \tilde{E}[\beta_{t,\tau}^{(1)} (P_\tau^{(1)} - K_\tau^{(1)})^+ - \beta_{t,\tau}^{(2)} (P_\tau^{(2)} - K_\tau^{(2)})^+] > 0\ .$$

To establish the last inequality, note that if $\tau = t$, the expectation above is $p^{(1)} - p^{(2)} > 0$. If $\tau > t$, $r^{(1)} < r^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} > \beta_{t,\tau}^{(2)}$, and $P_\tau^{(1)} \geq P_\tau^{(2)}$. Further, Corollary (7.4) implies that $K_\tau^{(1)} < K_\tau^{(2)}$. It follows now that $(P_\tau^{(1)} - K_\tau^{(1)})^+ \geq (P_\tau^{(2)} - K_\tau^{(2)})^+$, and $P_\tau^{(1)} \geq P_\tau^{(2)} > K_\tau^{(2)}$ with positive probability, so the integrand of the expectation is a.s. nonnegative and positive with positive probability.

3. Consider the cases $K_t = V_t$ and $K_t = k_t \wedge V_t$, and let $t < T$. Let $v^{(1)} < v^{(2)}$ be two possible values of the time t firm value, V_t. From Equation (22), $\forall s$ such that $t < s \leq T$, $v_s^{(1)} < v_s^{(2)}$. It follows that $K_\tau^{(1)} \leq K_\tau^{(2)}$, where τ is the optimal stopping time given that the state at time t is $P_t = p$ and $V_t = v^{(2)}$. The feasibility of τ as a stopping time for the state $P_t = p$ and $V_t = v^{(1)}$ implies that

$$f(p, v^{(1)}, t) - f(p, v^{(2)}, t) \geq \tilde{E}[\beta_{t,\tau}(P_\tau - K_\tau^{(1)})^+ - \beta_{t,\tau}(P_\tau - K_\tau^{(2)})^+] \geq 0 \; .$$

In the case of the pure default option, $K_t = V_t$, the last inequality is strict.

4. We let $p^{(1)} > p^{(2)}$ and prove that $f(p^{(2)}, v, t) - f(p^{(1)}, v, t) \geq p^{(2)} - p^{(1)}$. Let $r^{(1)} < r^{(2)}$ denote the time t interest rates corresponding to the two possible values for the time t bond price, $p^{(1)}$ and $p^{(2)}$, respectively. Let τ be the optimal stopping time for $p^{(1)}$. Then τ is a feasible stopping time for $p^{(2)}$ as well.

$$\begin{aligned}
f(p^{(2)}, v, t) - f(p^{(1)}, v, t) &\geq \tilde{E}[\beta_{t,\tau}^{(2)}(P_\tau^{(2)} - K_\tau^{(2)})^+] - \tilde{E}[\beta_{t,\tau}^{(1)}(P_\tau^{(1)} - K_\tau^{(1)})^+] \\
&= \tilde{E}[\beta_{t,\tau}^{(2)}(P_\tau^{(2)} - K_\tau^{(2)})^+ - \beta_{t,\tau}^{(1)}(P_\tau^{(1)} - K_\tau^{(1)})] \cdot 1_{(P_\tau^{(1)} > K_\tau^{(1)})} \qquad (30) \\
&\geq \tilde{E}[\beta_{t,\tau}^{(2)}(P_\tau^{(2)} - K_\tau^{(2)}) - \beta_{t,\tau}^{(1)}(P_\tau^{(1)} - K_\tau^{(1)})] \cdot 1_{(P_\tau^{(1)} > K_\tau^{(1)})} \\
&= \tilde{E}[\beta_{t,\tau}^{(2)} P_\tau^{(2)} - \beta_{t,\tau}^{(1)} P_\tau^{(1)}] \cdot 1_{(P_\tau^{(1)} > K_\tau^{(1)})} + \\
&\quad\; \tilde{E}[\beta_{t,\tau}^{(1)} K_\tau^{(1)} - \beta_{t,\tau}^{(2)} K_\tau^{(2)}] \cdot 1_{(P_\tau^{(1)} > K_\tau^{(1)})} \\
&\geq \tilde{E}[\beta_{t,\tau}^{(2)} P_\tau^{(2)} - \beta_{t,\tau}^{(1)} P_\tau^{(1)}] \cdot 1_{(P_\tau^{(1)} > K_\tau^{(1)})} \qquad (31) \\
&\geq \tilde{E}[\beta_{t,\tau}^{(2)} P_\tau^{(2)} - \beta_{t,\tau}^{(1)} P_\tau^{(1)}] \qquad (32) \\
&\geq p^{(2)} - p^{(1)}. \qquad (33)
\end{aligned}$$

Equation (30) follows from the fact that $r^{(1)} < r^{(2)} \Rightarrow P_\tau^{(1)} \geq P_\tau^{(2)}$ (Corollary 7.2), and $K_\tau^{(1)} \leq K_\tau^{(2)}$ (Corollary 7.4) which in turn imply that $P_\tau^{(1)} \leq K_\tau^{(1)} \Rightarrow P_\tau^{(2)} \leq K_\tau^{(2)}$. Inequality (31) follows from the fact that $r^{(1)} < r^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} K_\tau^{(1)} \geq \beta_{t,\tau}^{(2)} K_\tau^{(2)}$ (Corollary 7.1 and Equation 22). Inequality (32) follows from the fact that $r^{(1)} < r^{(2)} \Rightarrow \beta_{t,\tau}^{(1)} P_\tau^{(1)} \geq \beta_{t,\tau}^{(2)} P_\tau^{(2)}$ (Corollary 7.3). Finally, inequality (33) follows from Lemma 7.1.

5. We let $v^{(2)} > v^{(1)}$ and prove that $f(p, v^{(2)}, t) - f(p, v^{(1)}, t) \geq v^{(1)} - v^{(2)}$. Let τ be the optimal stopping time for $v^{(1)}$. Then τ is a feasible stopping time for $v^{(2)}$ as well.

$$\begin{aligned}
f(p, v^{(2)}, t) - f(p, v^{(1)}, t) &\geq \tilde{E}[\beta_{t,\tau}(P_\tau - K_\tau^{(2)})^+] - \tilde{E}[\beta_{t,\tau}(P_\tau - K_\tau^{(1)})^+] \\
&= \tilde{E}[\beta_{t,\tau}(P_\tau - K_\tau^{(2)})^+ - \beta_{t,\tau}(P_\tau - K_\tau^{(1)})] \cdot \\
&\quad 1_{(P_\tau > K_\tau^{(1)})} \qquad (34) \\
&\geq \tilde{E}[\beta_{t,\tau}(P_\tau - K_\tau^{(2)}) - \beta_{t,\tau}(P_\tau - K_\tau^{(1)})] \cdot 1_{(P_\tau > K_\tau^{(1)})} \\
&= \tilde{E}[\beta_{t,\tau}(K_\tau^{(1)} - K_\tau^{(2)})] \cdot 1_{(P_\tau > K_\tau^{(1)})} \\
&\geq \tilde{E}[\beta_{t,\tau}(K_\tau^{(1)} - K_\tau^{(2)})] \qquad (35) \\
&\geq \tilde{E}[\beta_{t,\tau}(V_\tau^{(1)} - V_\tau^{(2)})] \qquad (36) \\
&\geq \tilde{E}[e^{-\int_0^\tau \delta_u du} \cdot \beta_{t,\tau}(V_\tau^{*(1)} - V_\tau^{*(2)})] \\
&\geq \tilde{E}[\beta_{t,\tau}(V_\tau^{*(1)} - V_\tau^{*(2)})] \qquad (37) \\
&= v^{(1)} - v^{(2)}.
\end{aligned}$$

Inequalities (34) and (35) follow from the fact that $v^{(2)} > v^{(1)} \Rightarrow K_\tau^{(2)} \geq K_\tau^{(1)}$. Note that inequality (36) holds for all three specifications of K_t: $K_t = k_t$, $K_t = V_t$, or $K_t = k_t \wedge V_t$.□

Proof of Proposition 2.1 The first inequality is obvious. We establish the second inequality as follows.

$$f_{CD}(p, v, t) = \sup_{t \leq \tau \leq T} \frac{1}{\beta_t} \tilde{E}[\beta_\tau (P_\tau - k_\tau \wedge V_\tau)^+ | \mathcal{F}_t] \qquad (38)$$

$$= \sup_{t\le\tau\le T} \frac{1}{\beta_t}\tilde{E}[\beta_\tau((P_\tau - k_\tau)^+ \vee (P_\tau - V_\tau)^+)|\mathcal{F}_t] \quad (39)$$

$$\le \sup_{t\le\tau\le T} \frac{1}{\beta_t}\tilde{E}[\beta_\tau((P_\tau - k_\tau)^+ + (P_\tau - V_\tau)^+)|\mathcal{F}_t] \quad (40)$$

$$\le \sup_{t\le\tau\le T} \frac{1}{\beta_t}\tilde{E}[\beta_\tau(P_\tau - k_\tau)^+|\mathcal{F}_t] + \sup_{t\le\tau\le T} \frac{1}{\beta_t}\tilde{E}[\beta_\tau(P_\tau - V_\tau)^+|\mathcal{F}_t] \quad (41)$$

$$= f_C(p,v,t) + f_D(p,v,t)\ .\square \quad (42)$$

Proof of Theorem 3.1 Suppose $p^{(1)} \in U_{(v,t)}$ and $p^{(1)} > p^{(2)}$. We show that $p^{(2)} \in U_{(v,t)}$. From the call delta inequality, we have

$$f(p^{(2)},v,t) - f(p^{(1)},v,t) \ge p^{(2)} - p^{(1)}. \quad (43)$$

Thus,

$$\begin{aligned} f(p^{(2)},v,t) &\ge f(p^{(1)},v,t) + p^{(2)} - p^{(1)} \\ &> (p^{(1)} - K_t)^+ + p^{(2)} - p^{(1)} \\ &\ge (p^{(1)} - K_t) + p^{(2)} - p^{(1)} \\ &= p^{(2)} - K_t. \end{aligned} \quad (44)$$

From part 1 of Theorem 2.1, $f(p^{(2)},v,t) > 0$, so

$$f(p^{(2)},v,t) > (p^{(2)} - K_t)^+.$$

It follows that $U_{(v,t)}$ is of the form $(0, b(v,t))$. Finally, $b(v,t) > K_t$ must hold. Otherwise $f(b(v,t),v,t) = 0$, a contradiction.□

Proof of Theorem 3.2

1. Suppose $v^{(1)} \in U_{D(p,t)}$ and $v^{(1)} < v^{(2)}$. We show that $v^{(2)} \in U_{D(p,t)}$. From the put delta inequality,

$$f(p,v^{(2)},t) \ge f(p,v^{(1)},t) + v^{(1)} - v^{(2)} > (p - v^{(1)})^+ + v^{(1)} - v^{(2)} \ge p - v^{(2)}. \quad (45)$$

From part 1 of Theorem 2.1, $f(p,v^{(2)},t) > 0$, so

$$f(p,v^{(2)},t) > (p - v^{(2)})^+.$$

It follows that $U_{D(p,t)}$ is of the form $(v_D(p,t), \infty)$. Finally, $v_D(p,t) < p$ must hold. Otherwise $f(p, v_D(p,t), t) = 0$, a contradiction.

2. First, suppose it is optimal not to default at $v^{(1)}$ and $v^{(1)} < v^{(2)}$. We show that it is optimal not to default at $v^{(2)}$ as well. From the put delta inequality, we have

$$\begin{aligned} f(p, v^{(2)}, t) &\geq f(p, v^{(1)}, t) + v^{(1)} - v^{(2)} \\ &> (p - v^{(1)} \wedge k_t)^+ + v^{(1)} - v^{(2)} \\ &\geq (p - v^{(1)}) + v^{(1)} - v^{(2)} \\ &= p - v^{(2)}. \end{aligned}$$

From part 1 of Theorem 2.1, $f(p, v^{(2)}, t) > 0$, so

$$f(p, v^{(2)}, t) > (p - v^{(2)})^+.$$

It follows that it cannot be optimal to default at $v^{(2)}$. Note that it must be optimal to default at $v = 0$. Thus, there exists a critical value $v_{CD}^D(p,t)$ such that it is optimal to default $\forall v, v \leq v_{CD}^D(p,t)$. Further, $v_{CD}^D(p,t) < p$ must hold. Otherwise $f(p, v_{CD}^D(p,t), t) = 0$, a contradiction. In addition, $v_{CD}^D(p,t) \leq k_t$ must hold. Otherwise, there would exist a firm value greater than k_t at which it is optimal to default, which is impossible.

Next, we show that if it is optimal to call at $v^{(1)}$, and $v^{(1)} < v^{(2)}$, then it is optimal to call at $v^{(2)}$ as well. Note that $k_t \leq v^{(1)}$ must hold. Now, on one hand, $f(p, v^{(2)}, t) \geq g(p, v^{(2)}, t) = p - k_t \wedge v^{(2)} = p - k_t$. On the other hand, from part 3 of Theorem 2.1, $f(p, v^{(2)}, t) \leq f(p, v^{(1)}, t) = p - k_t$. Thus, if there exists any firm value at which it is optimal to call, then there exists a critical value $v_{CD}^C(p,t) \geq k_t$ such that it is optimal to call $\forall v, v \geq v_{CD}^C(p,t)$. □

Proof of Theorem 3.3

1. Firm value is not a state variable for the pure callable stopping problem.

2. Suppose $0 < p < b_D(v^{(1)}, t)$. It suffices to show that $p < b_D(v^{(2)}, t)$. From the put delta inequality, we have

$$f(p, v^{(2)}, t) \geq f(p, v^{(1)}, t) + v^{(1)} - v^{(2)} > p - v^{(1)} + v^{(1)} - v^{(2)} = p - v^{(2)}. \quad (46)$$

3. Suppose $v > v_D(p_2, t)$. It suffices to show that $v > v_D(p_1, t)$. From the call delta inequality, we have

$$f(p_1, v, t) \geq f(p_2, v, t) + p_1 - p_2 > (p_2 - v)^+ + p_1 - p_2 \geq p_1 - v. \; \square \quad (47)$$

4. The proof is essentially the same as that in part 2.

5. Let $v \leq k_t$ and $p < b_D(v, t)$. Then $f_{CD}(p, v, t) \geq f_D(p, v, t) > p - v = p - v \wedge k_t$, so $p < b_{CD}(v, t)$.

6. Suppose $0 < p < b_{CD}(v^{(2)}, t)$. It suffices to show $p < b_{CD}(v^{(1)}, t)$. We have

$$f(p, v^{(1)}, t) \geq f(p, v^{(2)}, t) > g(p, v^{(2)}, t) = (p - k_t)^+ = g(p, v^{(1)}, t). \quad (48)$$

7. Let $v > k_t$ and $p < b_C(v, t)$. Then $f_{CD}(p, v, t) \geq f_C(p, t) > p - k_t = p - v \wedge k_t$. So $p < b_{CD}(v, t)$.$\square$

Appendix 2: Numerical Implementation

Nelson and Ramaswamy (1990) show how to use binomial processes to approximate a general class of single-factor diffusions. To extend their analysis to multi-factor diffusion models, we first transform the state variables into new diffusion processes that are uncorrelated and have constant volatility. Then we construct a recombining, two-dimensional binomial lattice for the resulting orthogonalized diffusions. Finally, we transform the lattice for the orthogonalized state variables into a lattice for the original variables and price the callable and defaultable bonds using backward induction. This appendix describes the construction of the two-dimensional binomial lattice. Other papers illustrating the implementation of bivariate diffusions are Boyle, Evnine and Gibbs (1989), Hilliard, Schwartz and Tucker (1996), who consider lognormal processes, and Hull and White (1994-I, 1994-II, 1996), Ho, Stapleton and Subrahmanyam (1995), and Peterson, Stapleton and Subrahmanyam (1998), who consider two-factor term structure models.

To orthogonalize the interest rate and firm value processes r and V described in equations (1) and (3), first let $G_t = \frac{ln(V_t)}{\phi}$ and $H_t = \frac{2\sqrt{r_t}}{\gamma}$. Then, by Ito's Lemma,

$$\begin{aligned} dG_t &= \mu_t dt + d\tilde{W}_t \quad , \quad \mu_t = \frac{r_t - \delta_t - \frac{\phi^2}{2}}{\phi} \text{ , and} \\ dH_t &= \nu_t dt + d\tilde{Z}_t \quad , \quad \nu_t = \frac{\alpha - \frac{\gamma^2}{4} + \beta r_t}{\gamma\sqrt{r_t}} \ . \end{aligned} \tag{49}$$

Second, let $X_t = G_t$ and $Y_t = \frac{1}{\sqrt{1-\rho^2}}(-\rho G_t + H_t)$. Then X and Y are diffusions with unit instantaneous variance and zero cross-variation. The drift of X is $\mu_t^+ = \mu_t$ and the drift of Y is $\mu_t^- = \frac{1}{\sqrt{1-\rho^2}}(-\rho\mu_t + \nu_t)$.

The inverse transformation to obtain r_t and V_t from X_t and Y_t are

$$V_t = e^{\phi X_t} \quad , \quad r_t = \left[\frac{\gamma}{2} \cdot (\sqrt{1-\rho^2}Y_t + \rho X_t)\right]^2. \tag{50}$$

To get a lattice for r and V, we apply this inverse transformation at each node of the lattice for X and Y.

To construct a recombining, two-dimensional binomial lattice for the variables X and Y, we divide the time-interval $[0, T]$ into N equal intervals of length Δt. From a node (X_t, Y_t) at time t, the lattice evolves to four nodes, (X_t^+, Y_t^+), (X_t^+, Y_t^-), (X_t^-, Y_t^+), and (X_t^-, Y_t^-), where

$$\begin{aligned} X_t^+ &= X_t + (2k_1+1)\sqrt{\Delta t} \quad , \quad X_t^- = X_t + (2k_1-1)\sqrt{\Delta t}. \\ Y_t^+ &= Y_t + (2k_2+1)\sqrt{\Delta t} \quad , \quad Y_t^- = Y_t + (2k_2-1)\sqrt{\Delta t} \ , \end{aligned} \tag{51}$$

and k_1 and k_2 are integers such that

$$(2k_1-1)\sqrt{\Delta t} \le \mu_t^+ \Delta t \le (2k_1+1)\sqrt{\Delta t}, \tag{52}$$

$$(2k_2-1)\sqrt{\Delta t} \le \mu_t^- \Delta t \le (2k_2+1)\sqrt{\Delta t}. \tag{53}$$

The four nodes have associated risk-neutral probabilities $pq, p(1-q), (1-p)q$, and $(1-p)(1-q)$, respectively. The probabilities, p, of an up-jump in X_t process, and q, of an up-jump in Y_t process, are picked to ensure the right first moments at the node (X_t, Y_t):

$$p \;=\; \tfrac{1}{2} + \tfrac{\mu_t^{+}\sqrt{\Delta t}}{2} - k_1 \quad , \quad q \;=\; \tfrac{1}{2} + \tfrac{\mu_t^{-}\sqrt{\Delta t}}{2} - k_2 \; . \tag{54}$$

Equations (52) and (53) ensure that the probabilities are between 0 and 1. While the first moment of the process (X, Y) is matched exactly by the scheme above, the second moment is approximated with an error that is $O(\Delta t)$. The two-factor binomial process converges in distribution to the original continuous-time process as $\Delta t \to 0$.

To make the lattice for each state variable recombine, the variable can only move an integral number of increments $\sqrt{\Delta t}$, as equation (51) indicates. When the drift terms μ_t^{+} and μ_t^{-} are large in magnitude, for instance, at low interest rates when the speed of mean reversion is high, multiple jumps, that is, nonzero k_1 or k_2, occur. However, the lattice for each variable has only $n+1$ nodes at each time $n \cdot \Delta t$, so an up or down move from any node at time $(n-1) \cdot \Delta t$ must lead to one of the $n+1$ nodes at time $n \cdot \Delta t$. Therefore, the moves described in equations (52) and (53) require that Δt be sufficiently small. The numerical examples employ 35 to 40 time steps per year. We check the convergence by matching the price of a zero-coupon bond maturing at T, which can be calculated analytically, and by matching the price of a European default option on the zero-coupon bond with an expiration at T, under the two-factor specification, which can be calculated using Monte Carlo simulation.

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Table 1: Price and Sensitivity Measures for Benchmark High Grade and Junk Bond

	High grade (6.25%-coupon)				Junk (10.25%-coupon)			
	Host	C	D	CD	Host	C (called)	D	CD
Embedded option value		2.31	4.99	5.98		17.09	12.79	18.76
Bond price	99.81	97.50	94.82	93.83	117.10	100.00	104.31	98.34
Issuer's equity value			48.04	49.03			13.34	19.31
Yield to maturity (%)	6.30	6.85	7.50	7.75	6.28		9.19	10.69
Yield spread (bps)		55	120	145			291	442
Duration	4.29	3.45	3.59	3.40	4.01		2.45	1.32
Bond beta w.r.t. firm			0.16	0.14			0.38	0.21
Equity beta w.r.t firm			2.67	2.65			5.89	5.05
Equity Δ w.r.t. firm			0.90	0.91			0.67	0.83
One-factor bond Δ w.r.t. host		0.78	0.80	0.74			0.54	0.26
Two-factor bond Δ w.r.t. host		0.78	0.89	0.82			0.81	0.31
One-factor bond Δ w.r.t. equity			0.10	0.09			0.46	0.20
Two-factor bond Δ w.r.t. equity			0.12	0.10			0.49	0.20

A two-factor binomial lattice approximates the interest rate process, $dr_t = (\alpha + \beta r_t)dt + \gamma\sqrt{r_t}d\tilde{Z}_t$, and the firm value process, $\frac{dV_t}{V_t} = (r_t - \delta_t)dt + \phi d\tilde{W}_t$, with $d\langle \tilde{W}, \tilde{Z}\rangle_t = \rho dt$. The parameter values are $\alpha = 0.034$, $\beta = -0.5$, $\gamma = 0.10$, $\delta_t \equiv 0$, $\phi = 0.20$, and $\rho = 0$. The base case interest rate is $r_0 = 5\%$. All option, bond, equity, and firm values are in percent of bond par value. The base case firm values are $V_0 = 143$ for the high grade bond and $V_0 = 118$ for the junk bond. Host refers to the noncallable defaultable bond, C to the pure callable bond, D to the pure defaultable bond, and CD to the callable defaultable bond. All bonds have a maturity of $T = 5$ years. The callable bonds are currently callable at par. Call and default rules maximize option values. The durations, betas, and deltas with respect to firm value are calculated using a perturbation technique. The one-factor bond deltas are hedge ratios that make the one-factor hedged portfolio uncorrelated with the hedging instrument, either host bond or equity. The two-factor bond deltas are hedge ratios in a two-factor hedge using both host bond and equity as hedging instruments.

Table 2: Credit Spreads with Exogenous Default Rules or Constant Interest Rates

	High grade bonds		Junk bonds	
	Noncallable	Callable	Noncallable	Callable
Optimal default rule, stochastic interest rates, zero correlation, zero asset payout rate				
	120	145	291	442
Default when firm value falls to V^*, stochastic interest rates, zero correlation, zero asset payout rate				
V^* = bond par value, 100	21	56	247	400
V^* too low, 30 for high grade, 40 for junk	7	58	54	Called
Stochastic interest rates, zero correlation, positive asset payout rate δ				
Optimal default rule	252	265	521	576
Default when firm value falls to $V^* = c/\delta$,	207	221	373	439
c/δ = 80 for high grade, 95 for junk				
Optimal default rule, zero asset payout rate				
Constant interest rates	115	115	285	441
Stochastic interest rates, correlation = -0.5	33	75	157	386
Stochastic interest rates, correlation = 0	120	145	291	442
Stochastic interest rates, correlation = 0.5	228	235	427	507

A firm has outstanding a five-year bond, either noncallable or callable at par. All firm values in the table are expressed in percent of bond par value. In the high grade case, initial firm value is 143 and the bond coupon rate is $c = 6.25\%$. In the junk case, initial firm value is 118 and the bond coupon rate is c = 10.25%. The firm value process is $\frac{dV_t}{V_t} = (r_t - \delta_t)dt + \phi d\tilde{W}_t$, with $\phi = 0.20$. In the case of stochastic interest rates, the interest rate process is $dr_t = (\alpha + \beta r_t)dt + \gamma\sqrt{r_t}d\tilde{Z}_t$, with $\alpha = 0.034$, β = -0.5, γ = 0.10 and $r_0 = 5\%$. The correlation described in the table is ρ, where $d\langle\tilde{W}, \tilde{Z}\rangle_t = \rho dt$. Each risky bond has a corresponding noncallable, riskless host bond with the same coupon and maturity. In the case of constant interest rates, r equals the yield on the host bond from the stochastic interest case: 6.30% for the high grade bond and 6.28% for the junk bond.

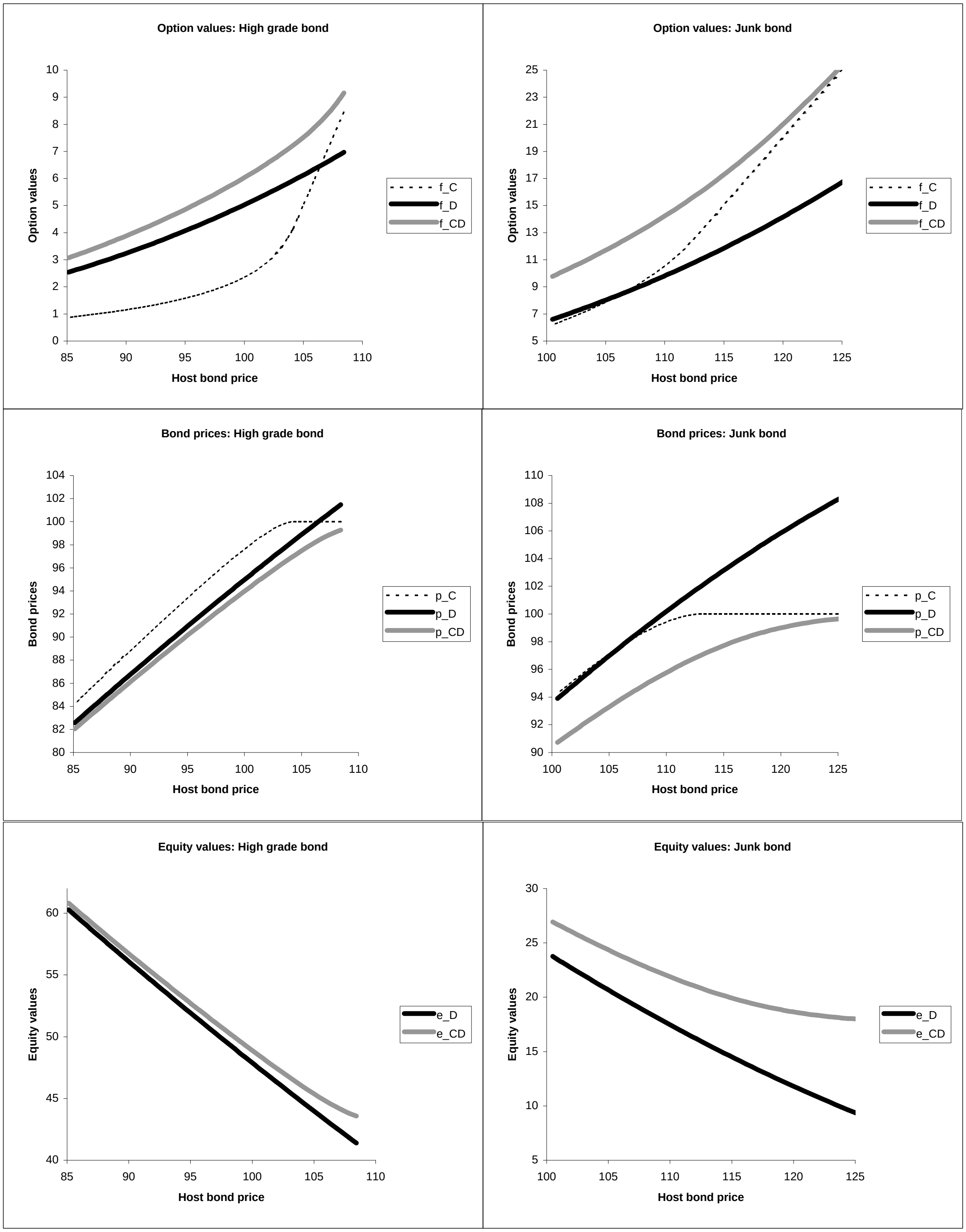

Figure 1
Option, bond, and equity values vs. host bond price

C refers to the pure callable bond, D to the pure defaultable, and CD to the callable defaultable. Values are generated using a two-factor binomial lattice that approximates a CIR interest rate process and a constant volatility firm value process. Call and default rules maximize equity values. All bonds have maturity of 5 years. The callable bonds are currently callable at par. The high grade bond has a 6.25% coupon and the issuer's firm value is 143. The junk bond has a 10.25% coupon and the issuer's firm value is 118.

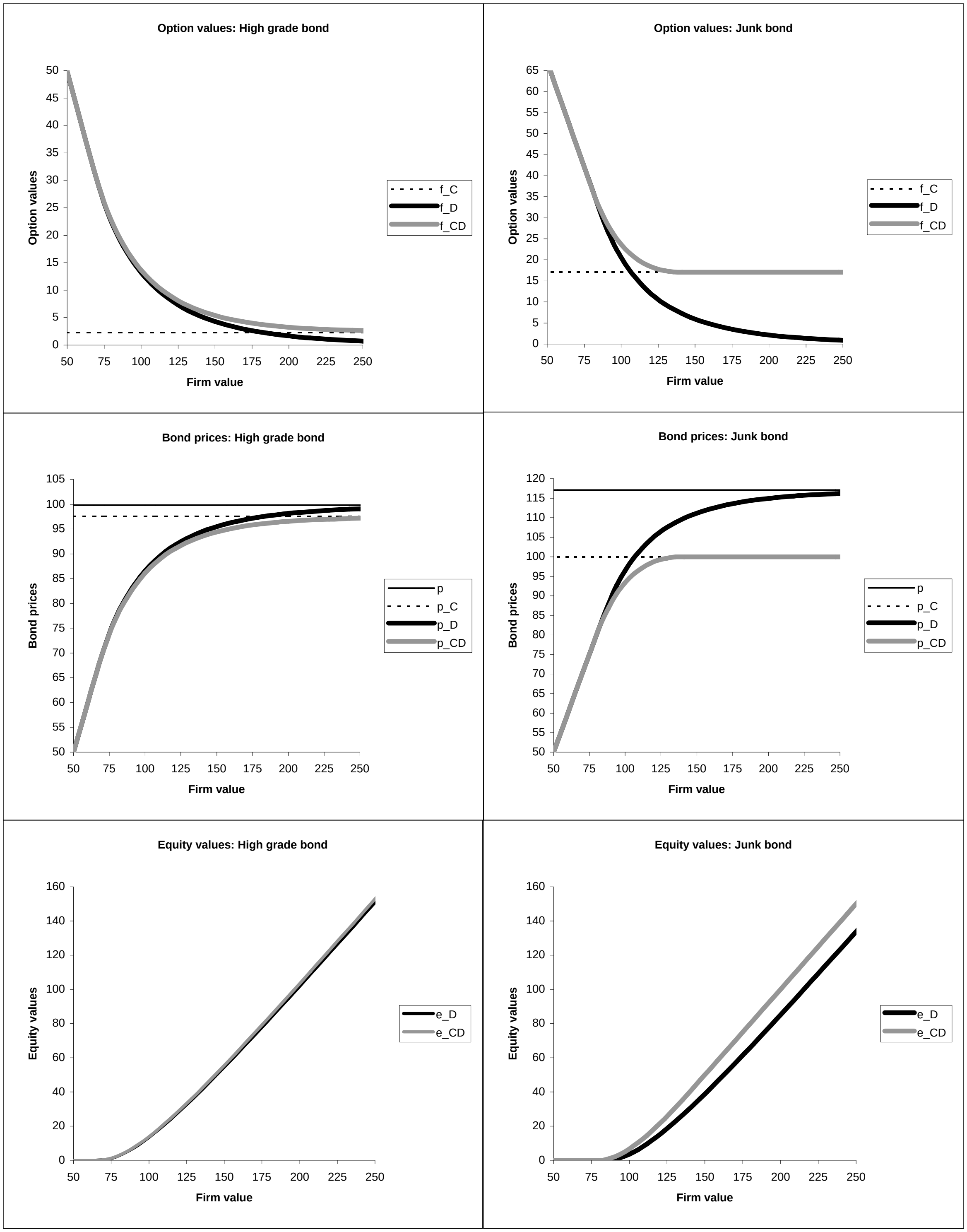

Figure 2
Option, bond, and equity values vs. firm value

C refers to the pure callable bond, D to the pure defaultable, and CD to the callable defaultable. Values are generated using a two-factor binomial lattice that approximates a CIR interest rate process and a constant volatility firm value process. Call and default rules maximize equity values. All bonds have maturity of 5 years. The callable bonds are currently callable at par. The high grade bond has a 6.25% coupon and the junk bond has a 10.25% coupon. The instantaneous interest rate is 5%.

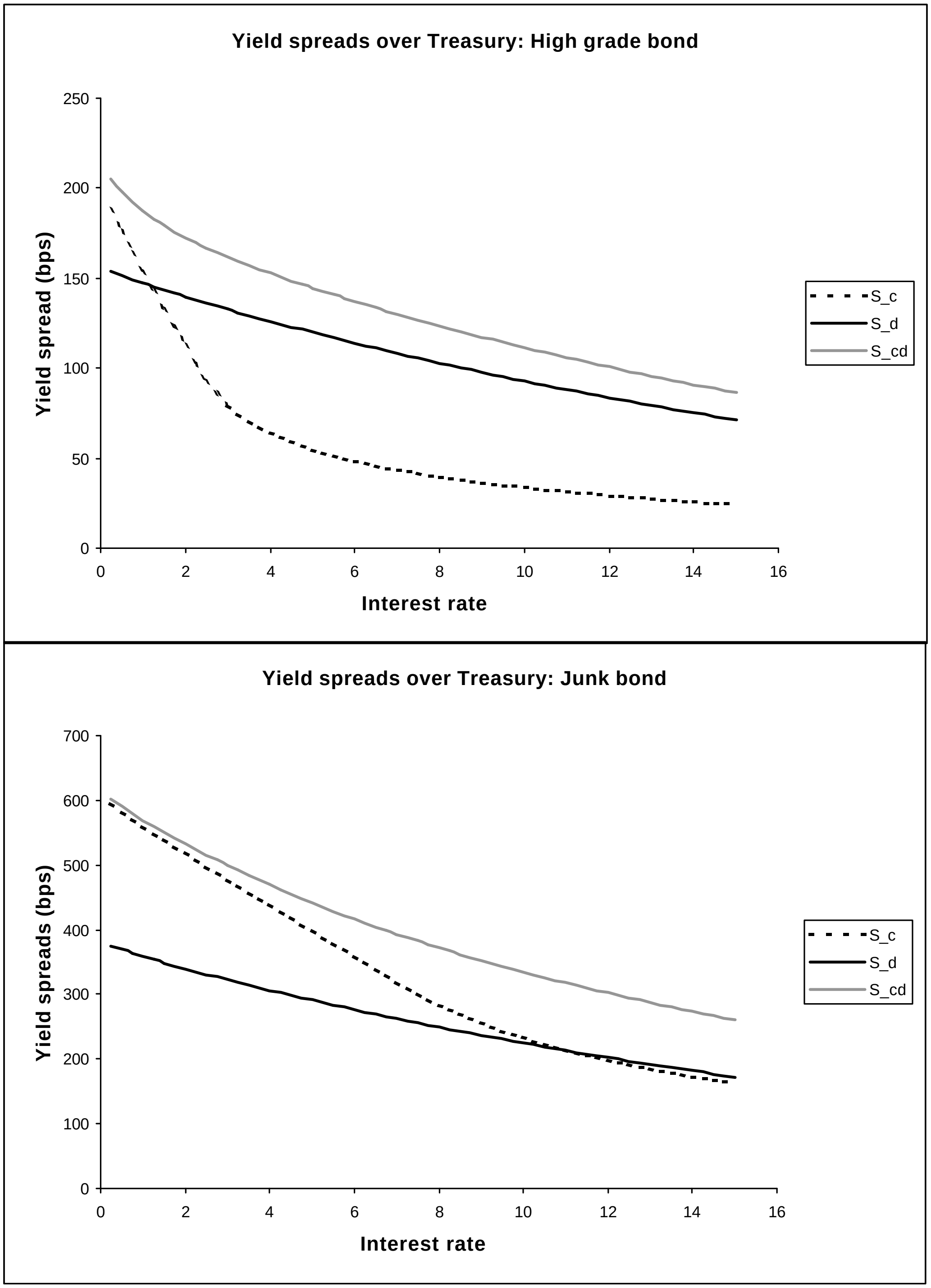

Figure 3

Yield spreads over Treasury

C refers to the pure callable bond, D to the pure defaultable, and CD to the callable defaultable. Bond prices and yields are generated using a two-factor binomial lattice that approximates a CIR interest rate process and a constant volatility firm value process. All bonds have maturity of 5 years. The callable bonds are currently callable at par. The high grade bond has a 6.25% coupon and the issuer's firm value is 143. The junk bond has a 10.25% coupon and the issuer's firm value is 118.

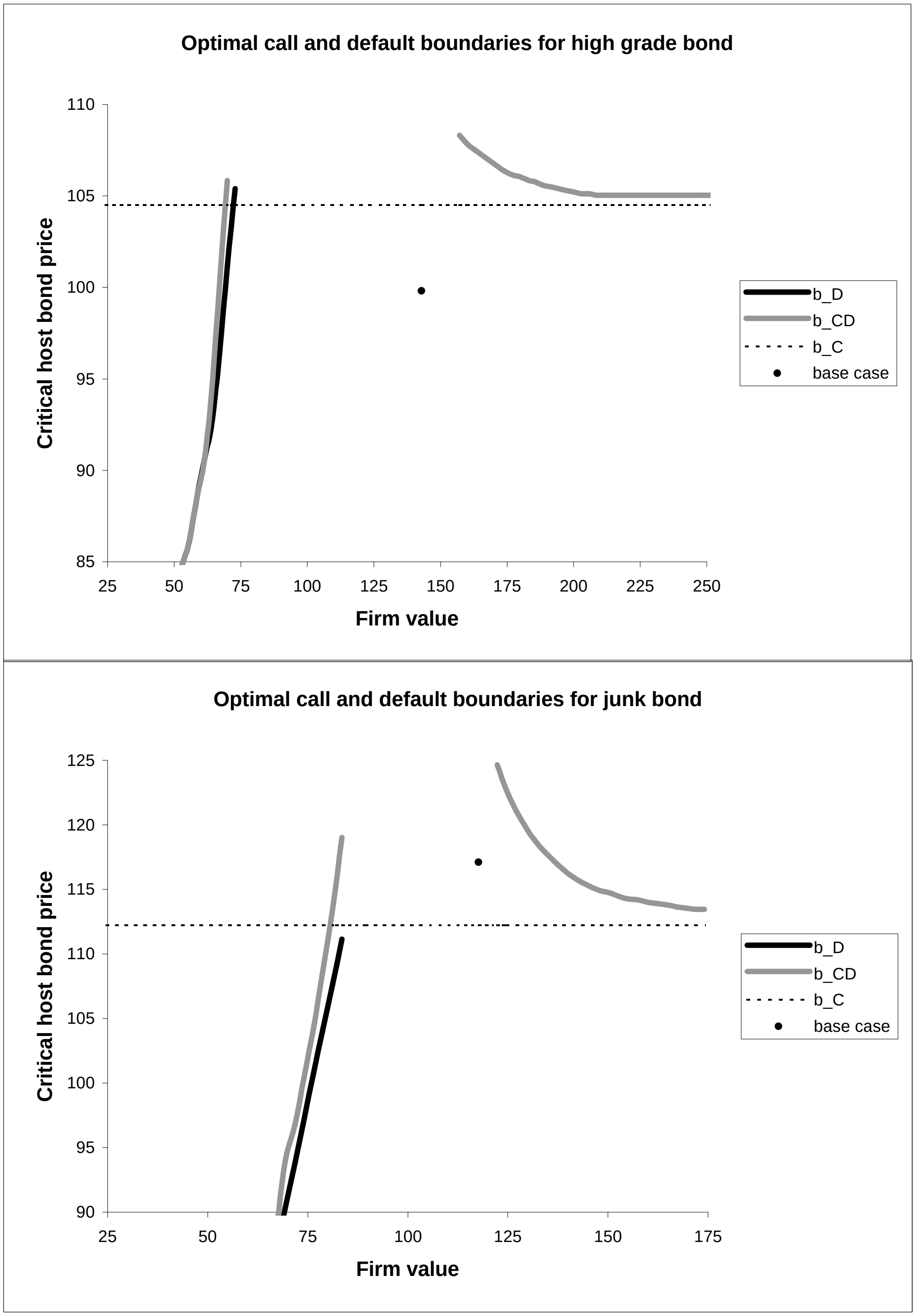

Figure 4
Optimal call and default boundaries
C refers to the pure callable bond, D to the pure defaultable, and CD to the callable defaultable. For host bond prices below the boundary, the issuer continues to service the bond. For host bond prices at or above the boundary, the issuer calls or defaults. These exercise rules maximize the value of issuer's equity, generated using a two-factor binomial lattice that approximates 5 years. The callable bonds are currently callable at par. The high grade bond has a 6.25% coupon and the junk bond has a 10.25% coupon. The bullets indicate base case host bond and firm values.

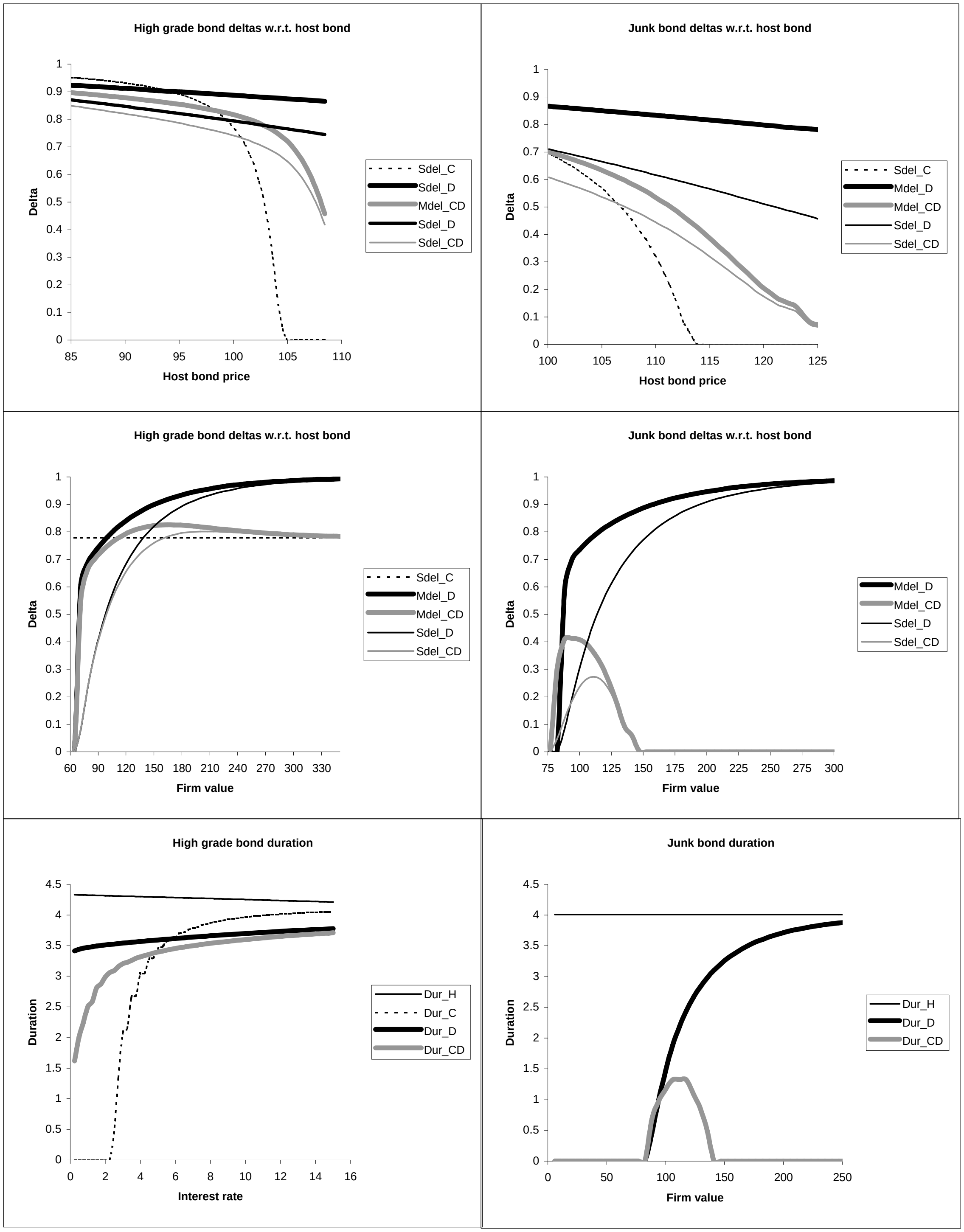

Figure 5

Interest rate risk measures: Deltas with respect to the host bond and durations

The number of host bonds necessary to hedge a given bond, such that the hedging error is uncorrelated with the hedging instruments. The single-factor deltas (Sdel) are for hedges using only the host bond. The multi-factor deltas (Mdel) are for hedges using both the host bond and the issuer's equity. C refers to the pure callable bond, D to the pure defaultable, and CD to the callable defaultable. Values are generated using a two-factor binomial lattice that approximates a CIR interest rate process and a constant volatility firm value process. Call and default rules maximize option values. All bonds have maturity of 5 years. The callable bonds are currently callable at par. The high grade bond has a 6.25% coupon and the issuer's firm value is 143 in the base case. The junk bond has a 10.25% coupon and the issuer's firm value is 118 in the base case. The base case instantaneous spot rate is 5%.

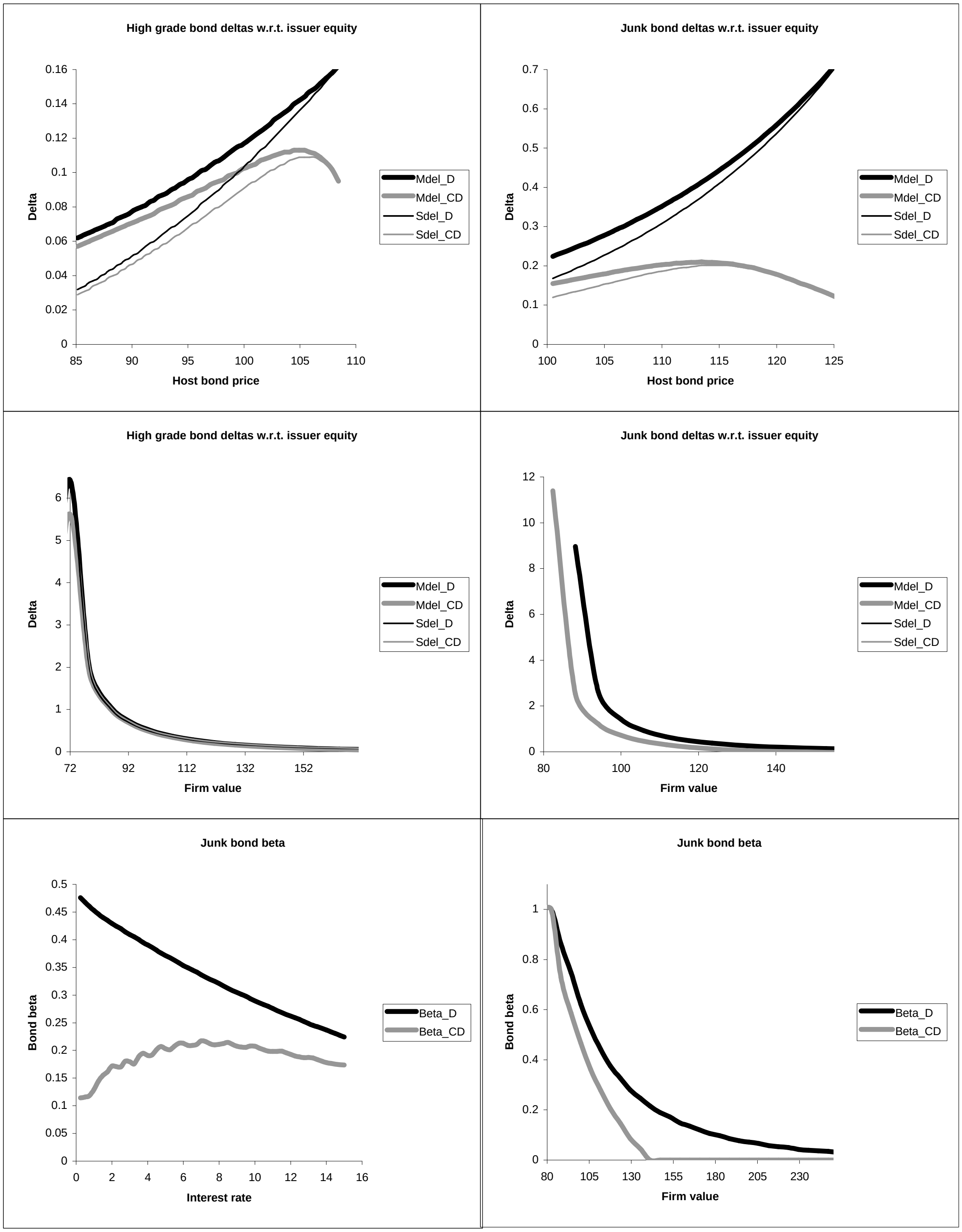

Figure 6

Equity risk measures: Deltas with respect to the issuer's equity and betas

The fraction of issuer's necessary to hedge a given bond, such that the hedging error is uncorrelated with the hedging instruments. The single-factor deltas (Sdel) are for hedges using only the issuer's equity. The multi-factor deltas (Mdel) are for hedges using both the host bond and the issuer's equity. C refers to the pure callable bond, D to the pure defaultable, and CD to the callable defaultable. Values are generated using a two-factor binomial lattice that approximates a CIR interest rate process and a constant volatility firm value process. Call and default rules maximize option values. All bonds have maturity of 5 years. The callable bonds are currently callable at par. The high grade bond has a 6.25% coupon and the issuer's firm value is 143 in the base case. The junk bond has a 10.25% coupon and the issuer's firm value is 118 in the base case. The base case instantaneous spot rate is 5%.