A Parimutuel Market Microstructure for Contingent Claims Trading

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Abstract

A parimutuel market microstructure for contingent claims trading is proposed and analyzed. A parimutuel microstructure is a call auction where relative equilibrium prices of contingent claims are endogenously determined using a specific mechanism. We propose a market microstructure incorporating parimutuel principles which provides for notional derivatives transactions, limit orders, and bundling of contingent claims across states. This microstructure will be used by Longitude Inc.'s clients to transact derivatives on economic statistics, weather, insurance losses and other types of risks. JPMorgan Chase and Deutsche Bank are some of the financial institutions that will be holding parimutuel auctions in early 2002.

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I. Introduction

The factors which motivate trade in asset markets—differences in information, risk preferences and endowments—have been widely studied in the literature on market microstructure. The early microstructure papers focused on the endowment factor in terms of dealer inventories. O'Hara (1995) provides a comprehensive summary. Models incorporating asymmetric information and risk preferences followed, beginning with the important work of Glosten and Milgrom (1985) and Kyle (1985).

A large literature also exists on security market microstructure design. For example, a substantial portion of the literature focuses on the specialist markets at the New York Stock Exchange. See, for example, Dupont (1995). Limit order book mechanisms, as a special type of continuous double auction, have also been studied extensively. See, for example, Handa and Schwartz (1996).

In this paper, we take a different methodological approach from the existing microstructure literature. We are interested in the question whether some microstructures might be more favorable for trading different types of assets. In the existing capital markets, the dominant market structure for all types of capital market assets are either order-driven continuous double auction mechanisms or quote-driven dealer mechanisms. These mechanisms are used whether the underlying asset types being transacted are equities, bonds, commodities, futures, or options. We propose a superior microstructure for trading contingent claims. The mechanism we are referring to is the parimutuel microstructure which is widely used for wagering.

Parimutuel principles were invented in late 19th century France by Pierre Oller as an alternative to the bookmaker syndicates that dominated French gaming at the time. The parimutuel mechanism supplanted bookmaker horse racing in the United States beginning in the 1920's and 1930's facilitated in large part by the invention of the automatic odds calculator (or "totelizator") by Harry Strauss.

As a market microstructure, the parimutuel mechanism has four distinguishing features: (1) the parimutuel mechanism is a call auction market rather than a continuous auction; (2) relative prices of contingent claims are equal to the relative aggregate cost of such claims; (3) the total amount paid for the contingent claims is exactly sufficient to pay for the contingent claims having a positive return, that is, the mechanism is self-funding and risk-neutral in the sense that the total premium paid for contingent claims is equal to the state contingent payouts for all contingent claims expiring "in-the-money; and (4) a unique set of endogenously determined prices is discovered.

While much empirical work has been done on the efficiency and information characteristics of parimutuel wagering,[1] not much has been done to ground the mechanism in the standard theory of market microstructure. We provide a first step by showing a foundational connection between parimutuel principles and the theory of market games. In Section I, we show that a parimutuel contingent claims market is a natural extension of a Shapley-Shubik market game for contingent claims.[2] We can therefore connect the parimutuel mechanism to the well-developed market games literature and show that a parimutuel mechanism is a viable mechanism for a contingent claims market with endogenous price formation. In Section II, we introduce a parimutuel market microstructure for trading contingent claims over a probability state space of contingent outcomes. We show that a parimutuel microstructure with notional claims, limit orders, and "claim bundling" across states has a unique price equilibrium. We also present a theorem which shows that all parimutuel mechanisms can be expressed as a solution to a general eigenvalue problem. Section III discusses the efficiency and no-arbitrage characteristics of the parimutuel microstructure we propose. Section IV contains concluding remarks. In the Appendix, we provide techniques for solving the equilibrium of the parimutuel trading mechanism proposed.

We believe the proposed microstructure to be a compelling alternative to the existing continuous time options and futures markets for derivatives securities. First, the microstructure proposed is a complete framework, in the sense that it addresses the valuation, liquidity, and information aspects of a market in a unified manner. Valuation is performed according to the parimutuel principle that the value of a contingent claim is based upon the relative premium investment in that claim to the total premium investment for all claims across the state space. We believe these simple valuation

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[1] See Haush, Lo, and Ziemba (1994)

[2] See Shapley and Shubik (1977).

principles are a compelling alternative to the widely known and researched continuous microstructure valuation techniques, which have originated with the continuous time microstructure framework of Black and Scholes (1973). The call auction aspect of the parimutuel microstructure, as well as the ability to aggregate and link the state contingent payouts within a state space in a unified manner, promotes the aggregation of liquidity. Implied state probabilities are a natural by-product of trading in the parimutuel microstructure. This provides a rich informational benefit compared to the existing continuous time options markets.

Second, we believe the microstructure is clearly superior for transacting illiquid risks for which there is no available or liquid underlying security or instrument in which to hedge. *The parimutuel mechanism requires no tradable underlying security* since all contingent claim premia are used to fund contingent claim payouts. There is therefore no requirement for hedging in a tradable underlying instrument.

Third, we believe that the parimutuel microstructure is superior to other call auction mechanisms for contingent claims since it provides unique and arbitrage-free contingent claim prices with the ability to flexibly bundle elemental contingent claims together within the same auction. These features promote not only aggregation and efficient use of liquidity, but also a highly transparent means of synthesizing or replicating arbitrary payout distributions from the elemental contingent claims.

II. Parimutuel Microstructure and Market Games

Parimutuel principles are widely used as an alternative to fixed odds gambling in which a bookmaker acts as a dealer by quoting fixed rates of return on specified wagers. A parimutuel game is conducted as a call auction in which odds are allowed to fluctuate during the betting period until the betting period is closed or the auction "called." The prices or odds of wagers are set based upon the relative amounts wagered on each risky outcome. In microstructure terms, wagering under parimutuel principles is characterized by (1) call auction, non-continuous trading; (2) riskless funding of claim payouts using the amounts paid for all of the claims during the auction; (3) special equilibrium pricing conditions requiring the relative prices of contingent claims equal the relative aggregate amounts wagered on such claims; and (4) endogenous determination of unique state prices.

In the theory of contingent claims markets, the self-funding and relative pricing features of a parimutuel system result from the guaranteed existence of a positive state price vector, *p*, which excludes arbitrage over the state space.[3] The vector p contains the prices for each elemental state outcome. We will show the existence of the positive state price vector combined with enforcing the equality of the aggregate payouts for each state are sufficient to guarantee that contingent claims are both self-funding and that the relative prices of claims are equal to the relative amounts paid for such claims. Assuming no transaction costs, and for purposes of this discussion, zero interest rates, the absence of arbitrage requires the following *normalization condition* on the state prices:

$$\begin{aligned} p^T e &= 1, \\ p &> 0 \end{aligned} \qquad \textbf{(1)}$$

where p is a strictly positive m-dimensional vector of state prices (probabilities), e is an m-dimensional unit vector, and superscript T is the familiar transpose operator. Multiplying by a vector y, an m-dimensional vector containing the aggregate state payouts for each state, yields the *riskless condition* that all payouts are identical across the states:

$$(y^T p)e = y. \qquad \textbf{(2)}$$

Since the left-hand side of (2) is a vector containing the aggregate premium investment, (2) states that the state contingent payout of each state is equal to the aggregate premium investment, i.e., that total amounts paid for all of the contingent claims are equal to the total contingent payouts. And since there is no arbitrage, the pricing system is linear, so that clearly:

$$\frac{p_i}{p_k} = \frac{y_i p_i}{y_k p_k} = \frac{(y^T p) p_i}{(y^T p) p_k} \qquad \textbf{(3)}$$

where p_i and y_i are the i-th elements of the vector p and y, respectively. This condition states that the relative prices of each fundamental state contingent claim is equal to the aggregate relative amounts paid for the respective claims.

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3 We employ the term state space to include the usual formalism, i.e., a set Ω contains an algebra of events, *F*, for which there exists a probability measure P: *F*→ [0, 1] satisfying P(Ø) = 0 and P(Ω) = 1 and for any disjoint events A and B:

$$P(A \cup B) = P(A) + P(B).$$

The triple (Ω, *F*, P) is called a probability state space, or "state space." See Duffie (1992), Appendix A.

In addition, parimutuel principles include a market structure for arriving at the equilibrium prices in which state prices are discovered endogenously via a call auction process and it is the endogenous nature of the price discovery which provides a fundamental connection to the existing contingent claims and market microstructure research.

The seminal paper of Arrow (1964) demonstrated the equivalence of a competitive exchange economy for contingent commodities with an economy which has a complete and competitive securities market and a spot market in the commodities. In this competitive analysis, the securities market has contingent claims prices which are fixed exogenously. Since prices are fixed, each agent's demand has a negligible effect on the price. Subsequent research has shown that this equivalence result depends crucially on the competitive nature of the securities markets. For example, Peck, Shell, and Spear (1992) show that if the securities market is modeled using a noncooperative market game with endogenous price formation, then the Arrow equivalence result no longer holds. See also Weyers (1999).

The market microstructure literature is largely concerned with endogenous price formation where each agent's demand has a potentially significant impact on the market price. Outside the finance literature, there exists a large body of research utilizing the theory of noncooperative market games to model endogenous price formation. An influential paper by Shapley and Shubik (1977) introduced a noncooperative market game for a market with commodities and fiat money but with no uncertainty. In the Shapley-Shubik market game ("SSMG"), each trader consigns his endowment of each commodity to a trading post dedicated to that commodity. Trade occurs with each trader bidding some of his fiat money to each trading post. When the trading period ceases, the equilibrium price of each commodity is the sum of all the bids in fiat money committed to a trading post divided by the total quantity of commodity consigned. Each trader receives an amount of goods resulting from his bid of fiat money equal to his bid divided by the equilibrium price. Shapley and Shubik (1977) and subsequent papers show that an interior Nash Equilibrium (NE) always exists and that the NE converges to a competitive equilibrium as the economy is replicated. See, for example, Powers, Shubik, and Yao (1994).

The SSMG framework has been applied to markets with uncertainty by Peck, Shell, and Spear (1992) and Weyers (1999) as indicated above. Our intent here is to analyze an SSMG market adapted to contingent claims over a state space, i.e., we are interested in the securities market microstructure which may be generally applicable to derivatives and other contingent claims markets. We first show that the SSMG market game with a credit policy restriction on selling is a parimutuel market microstructure. The credit policy, which is defined further below, requires that selling be done on a secured or collateralized basis.

***Proposition 1**: A Shapley-Shubik market game for contingent claims within a probability state space with secured selling is a parimutuel market.*

***Proof**:* The following notation is required:

n agents indexed i = 1, …, n;

m states indexed j = 1, …, m;

$w^i(o)$, initial wealth of agent i;

$w_j^i(f)$, final wealth of agent i in state j;

b_j^i, agent i's bid in dollars for state contingent claim j;

x_j^i, agent i's offer in dollars for insuring contingent claim j; and

p_j, price for state j.

First we define the Shapley-Shubik market model. In the Shapley-Shubik model, each trader makes bids and offers to each trading post, where each trading post corresponds to a contingent claim within a probability state space. As in the classical Shapley-Shubik market game, prices are equal to the ratio of total money bids divided by total commodity consignments or offers for each trading post. For a contingent claims market using the above notation, endogenous price formation therefore takes the following well-known functional form

$$p_j \equiv \frac{\sum_{i=1}^{n} b_j^i}{\sum_{i=1}^{n} x_j^i} . \qquad \textbf{(4)}$$

Each state contingent claim price is therefore the sum total of bids in units of money (e.g., dollars) divided by offers in units of money. The offers can be interpreted as sales of the contingent claim, or offers to payout 1 unit of state contingent insurance should the state corresponding to the trading post be realized.

Based upon the preceding notation, the budget constraint for agent i is therefore

$$w_j^i(f) = w^i(o) - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} x_j^i p_j + \frac{b_j^i}{p_j} - x_j^i, \quad \forall \mathrm{j} = 1, ..., \mathrm{m}. \tag{5}$$

We assume interest rates are zero and there is no production. Thus, the initial and final wealth in the economy are equal

$$\sum_{i=1}^{n} w_j^i(f) = \sum_{i=1}^{n} w^i(o) \rightarrow -\sum_{i=1}^{n}\sum_{j=1}^{m} b_j^i + \sum_{i=1}^{n}\sum_{j=1}^{m} x_j^i p_j + \sum_{i=1}^{n} \frac{b_j^i}{p_j} - \sum_{i=1}^{n} x_j^i = 0, \quad \forall \mathrm{j} = 1, ..., \mathrm{m}. \tag{6}$$

as implied from the definition of price p_j,

$$\sum_{i=1}^{n} \frac{b_j^i}{p_j} - \sum_{i=1}^{n} x_j^i = 0, \quad \forall \mathrm{j} = 1, ..., \mathrm{m}. \tag{7}$$

We refer to this condition as the *market clearing condition*. Summing over j yields the initial (i.e., at the time of premium settlement) market clearing condition that total premiums paid equal total premiums sold, or:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} b_j^i - \sum_{i=1}^{n}\sum_{j=1}^{m} x_j^i p_j = 0. \tag{8}$$

Since all the states comprise a state space, it is required that:

$$\sum_{j=1}^{m} p_j = \sum_{j=1}^{m} \frac{\sum_{i=1}^{n} b_j^i}{\sum_{i=1}^{n} x_j^i} = 1. \tag{9}$$

Clearly, nothing so far developed prevents sellers of claims (i.e., sellers of "insurance") from defaulting. To address the probability of default, we assume that the market imposes the following credit restriction on offers of notional insurance:

Define a credit policy as follows: Total offers of notional insurance for any state must be secured by at least the total premiums sold for all of the states, i.e.,

$$\sum_{i=1}^{n} x_j^i \le \sum_{i=1}^{n}\sum_{j=1}^{m} x_j^i p_j, \forall j. \tag{10}$$

Substituting from the market clearing condition, yields:

$$\sum_{i=1}^{n} \frac{b_j^i}{p_j} \le \sum_{i=1}^{n} \sum_{j=1}^{m} b_j^i, \quad \forall \mathrm{j} = 1, ..., \mathrm{m}. \qquad \textbf{(11)}$$

which yields

$$\frac{\sum_{i=1}^{n} b_j^i}{\sum_{i=1}^{n} \sum_{j=1}^{m} b_j^i} \le p_j, \quad \forall \mathrm{j} = 1, ..., \mathrm{m}. \qquad \textbf{(12)}$$

Since the states comprise a probability state space,

$$\sum_{j=1}^{m} p_j = 1. \qquad \textbf{(13)}$$

Thus, it must be the case that

$$\frac{\sum_{i=1}^{n} b_j^i}{\sum_{i=1}^{n} \sum_{j=1}^{m} b_j^i} = p_j, \quad \forall \mathrm{j} = 1, ..., \mathrm{m}, \qquad \textbf{(14)}$$

which states that the price of each state is equal to the total bids for that state divided by the total bids for all of the states. Thus, the equilibrium pricing condition for the Shapley-Shubik market game for contingent claims requires the relative prices of contingent claims to equal the relative aggregate bids for the respective claims. Since the SSMG is also a call auction market which is self-funding with endogenous price determination, the SSMG for contingent claims is parimutuel.

We can also interpret Proposition 1 in the following way. Each trader who makes an offer for a contingent claim (i.e., a sale of notional insurance) is required to post margin. The margin amount is equal to the premium proceeds. This is a standard practice at most options exchanges and is known as *premium margin*. Proposition 1 requires that the total amount of notional insurance on offer for any state cannot exceed the total premium margin deposited. At most options exchanges, an additional amount of margin related to the risk of the option sold is also required (oftentimes known as *additional margin* as is the case at Eurex Clearing A.G., the clearinghouse for the Eurex exchange). As no additional margin is required by Proposition 1, we interpret the credit policy to be not overly tight, especially as compared to existing margin mechanisms in use.

Proposition 2: *The credit policy constraint requiring the total notional offers of insurance for any state not exceed the total premiums sold can always be satisfied, i.e., it is never binding.*

Proof: It can easily be shown that any notional sale can be replicated through a purchase of complementary states within the state space over which claims are traded so that

$$\sum_{i=1}^{n} x_j^i \le \sum_{i=1}^{n}\sum_{j=1}^{m} x_j^i p_j = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j^i, \forall j. \quad \textbf{(15)}$$

Consider a notional sale where

$$x_k^i > 0 \text{ and } x_j^i = 0 \text{ for } j \neq k.$$

In this case, agent i sells a claim on state k and on no other state. We use the term *replicated sale* to denote the strategy of bidding on the complementary states to state k in the following way[4]:

$$b_k^i = 0 \text{ and } b_j^i = p_j x_j^i \text{ for } j \neq k.$$

The bid on the kth state of the replicated sale is 0, whereas bids on all other states are non-zero. To ensure the replication is available, we allow the trading post for each state-contingent claim to open with an arbitrarily small bid and offer, i.e.,

$$\varepsilon_j(b) \to 0, \ \ \varepsilon_j(x) \to 0, \forall \mathrm{j} = 1, ..., \mathrm{m},$$

where the arguments b and x indicate the small amounts of existing bids and offers allocated to each state, where these amounts are vanishingly small.[5] In equilibrium, the profits of a replicated sale are identical to those of the original notional sale,

$$\begin{aligned} w_k^i(f) &= w^i(o) - \sum_{j \neq k} p_j x_k^i = w_k^i(o) - x_k^i(1 - p_k), \\ w_{j\neq k}^i(f) &= w^i(o) - \sum_{j \neq k} p_j x_k^i + x_k^i = w^i(o) + x_k^i p_k, \end{aligned} \quad \textbf{(16)}$$

i.e., the final wealth from the replicated sale is identical to the original notional sale for each state. Any notional sale can therefore be replicated into a complementary bid which satisfies the credit policy, and therefore replicated sales are payout-achievable. We have yet to show that an equilibrium exists with such replication going on during the auction.

[4] We note that bidding on all of the states proportional to the price achieves the “autarky” strategy of effecting no change in each agent’s endowments. See Peck, Shell, and Spear (1992).

[5] These small liquidity amounts take the place of the usual SSMG convention that the quantity 0/0 owing to zero bids and offers is equal to 0.

We turn to this next, and show that any arbitrary number of replications has a fixed-point equilibrium.

***Proposition 3**: A unique parimutuel equilibrium exists with replicated sales which are used to satisfy the credit policy.*

***Proof**:* Consider a notional sale where

$$x_k^i > 0 \text{ and } x_j^i = 0 \text{ for } j \neq k.$$

As indicated above, the replicated sale strategy is

$$b_k^i = 0 \text{ and } b_j^i = p_j x_j^i \text{ for } j \neq k$$

such that the strategy bid for $j \neq k$ is as follows:

$$b_j^i = \frac{\sum_i^n b_j^i}{\sum_i^n \sum_j^m b_j^i} x_j^i = g(b_j^i). \quad \textbf{(17)}$$

By the Banach Fixed Point Theorem, there exists a fixed point strategy bid for the differentiable function $g(b_j^i)$ if there exists a constant $z < 1$ such that, for all b_j^i

$$\left| g'(b_j^i) \right| \leq z. \quad \textbf{(18)}$$

Differentiation of $g(b_j^i)$ yields:

$$g'(b_j^i) = \frac{\sum_i^n \sum_j^m b_j^i - \sum_i^n b_j^i}{(\sum_i^n \sum_j^m b_j^i)^2} x_j^i. \quad \textbf{(19)}$$

From the market clearing condition, above,

$$x_j^i \leq \sum_{i=1}^n x_j^i \leq \sum_i^n \sum_j^m b_j^i. \quad \textbf{(20)}$$

Together with the obvious:

$$\frac{\sum_i^n \sum_j^m b_j^i - \sum_i^n b_j^i}{(\sum_i^n \sum_j^m b_j^i)^2} < 1 \quad \textbf{(21)}$$

completes the proof. Uniqueness follows from the contraction property of the fixed point.

Proposition 4: *The SSMG and a parimutuel market have equivalent payouts and first-order optimality conditions.*

Proof: See Appendix II.

The first order conditions derived in the Appendix can be manipulated to derive

$$\frac{q_j^i u'(w_j^i(f))}{q_k^i u'(w_k^i(f))} = \frac{p_j}{p_k}\left\{\frac{1+\dfrac{b_j^i}{B_j^i}}{1+\dfrac{b_k^i}{B_k^i}}\right\}, \tag{22}$$

which shows that the ratio of expected marginal utilities is equal to the ratio of state contingent prices, i.e., the competitive Arrow-Debreu economy result, multiplied by the term in brackets. These are the same oligopoly conditions derived in Shapley-Shubik (1977). The oligopoly conditions depend on the ratio of the size of trader i's bid in each state to the total amount of bids in that state excluding trader i's bid. They therefore measure the market's ability to absorb trader i's bid strategy. As trader i's bid strategy becomes smaller relative to the total bids on each state, the market approaches the competitively optimal market.

III. Parimutuel Microstructure for Contingent Claims Trading

In this section we develop a contingent claims market with a parimutuel microstructure which may be readily used in the capital markets. Our primary goal is to design the market such that contingent claims may be traded in a manner familiar to market participants in existing derivatives markets. To this end, we have first required that all trading strategies be implementable with bids and offers of *notional* amounts of risky claims. In the Shapley-Shubik contingent claims market game of Section I, agents implement strategies with offers of notional insurance x_j^i and bids of premium dollars, b_j^i. Conventionally, however, derivatives contracts are based upon the notional amount to be bought or sold and not denominated in premium dollars. The purchaser of an option, say on the dollar-yen foreign exchange rate, will specify a desired size of the position in notional terms, e.g., 10 million dollars, rather than in terms of the amount of desired premium outlay. Second, we provide for trading strategies which allow for arbitrary ratios of notional payouts across states. We introduce the concept of *claim bundling* across states which allocates premium dollars to states in order to generate the requested ratios of notional payouts across the states. Third, we allow trading strategies to be implemented with *limit orders*, whereby a trader may specify a reservation price above (below) which the specified purchase (sale) of a given contingent claim will not be executed.

A. Development of the Parimutuel Microstructure: Definitions

In this section, we develop the concepts and mathematical notation needed for a parimutuel contingent claims microstructure in which trader strategies can be implemented with (1) a notional buy or sell order; (2) a vector of payout ratios corresponding to a range of states (claim bundling); and (3) a limit price. For the purpose of developing a general model, we first allow a trading strategy to include orders based upon notional as well as premium. We begin by defining the strategy space for our parimutuel microstructure. We find it convenient to define the strategy space in terms of *orders* for contingent claims. The structure for a given order i is defined to be:

$$s_i = \left\{ \begin{matrix} r_i, u_i, c_i, w_i, \delta_i \,\middle|\, r_i, u_i \geq 0; c_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,m-1}, c_{i,m}), \\ \max(c_{i,j}) = 1, 0 \leq c_{i,j} \leq 1; 0 \leq w_i \leq 1; \delta_i = \{-1, 1\} \end{matrix} \right\}. \quad \textbf{(23)}$$

Here order i comprises (1) a notional or premium amount request, denoted by r_i and u_i, respectively, expressed in terms of a valuable unit of account which, for the present discussion, will be assumed to be fiat money; (2) a state coverage vector specifying for which states payouts are desired and their corresponding ratios—the $c_{i,j}$ are normalized so that one of the elements is the "state numeraire"; (3) a limit price and (4) an indicator variable, δ_i, which specifies a –1 for a sale and 1 for a purchase of the states specified in c_i.

Let p be the m by 1 vector where p_j, the j-th element of p, is the executed premium per dollar payout for state j, j =1, 2, …, m. We refer to p_j as the *price of state j* or *the implied probability of state j*. We assume that all prices are positive,

$$p_j > 0 \text{ for } j = 1, 2, \ldots, m, \quad \textbf{(24)}$$

and they sum to 1,

$$\sum_{j=1}^{m} p_j = 1. \quad \textbf{(25)}$$

These assumptions guarantee that there is no riskless arbitrage from an autarky strategy.[6]

Let π_i denote the executed premium for the order with specified states c_i per dollar payout or *the price of order i*. We assume that

$$\pi_i \equiv c_i^T p \text{ for } i = 1, 2, \ldots, n. \tag{26}$$

This definition says that the price of any contingent claim specified in order i must equal the weighted sum of the state prices for which order i has specified payouts. We define x_i and v_i as the executed quantity for an order denominated in notional and premium terms respectively where

$$\begin{aligned} &0 \le x_i \le r_i, \ i = 1, ..., n, \\ &0 \le v_i \le u_i, i = 1, ..., n. \end{aligned} \tag{27}$$

Note that

$$\frac{v_i}{x_i} = \pi_i, i = 1, ..., n, \tag{28}$$

which states that the ratio of the executed premium amount to the executed payout amount is equal to π_i. Once π_i and x_i are determined, then v_i can be computed. Therefore, it must be the case that

$$0 \le x_i \le \frac{u_i}{\pi_i}, i = 1, ..., n. \tag{29}$$

We can therefore convert all premium-based orders into notional executions $x_i(p)$ since

$$x_i(p) = \frac{v_i}{\pi_i}, i = 1, ..., n, \tag{30}$$

where the dependence of x_i on p is implicit for any order based upon premium. Defining C to be an n x m state coverage matrix containing for each order i, i = 1, …, n, and each state j, j = 1, …, m, the requested payout coverage and ratios for each order, we define the total notional requested per state as follows:

$$y_j(p) = C^T \text{x}, i = 1, ..., n, \tag{31}$$

where y_j's dependence upon the state price vector p owes solely to the presence of premium-based orders.

[6] Technically, the sum of these prices should sum to the state price deflator from the time of premium settlement to the time of payout settlement. For simplicity, we maintain the assumption of Section I that interest rates are zero and the state price deflator is equal to one.

To simplify we note that from Section I, a notional sell order can be replicated into a buy order on states complementary to the states being sold with a corresponding limit price for the replicated buy order equal to one minus the limit sell price, i.e.,

$$s_i = \{r_i, c_i, w_i, -1\} = \{r_i, e - c_i, 1 - w_i, 1\} \quad \textbf{(32)}$$

where e is unit vector of dimension m. Similarly, a premium-based order can be transformed, as noted above, into a notional-based order with the requested notional for the premium orders allowed to depend upon the equilibrium prices

$$s_i = \{u_i, c_i, w_i, 1\} = \{r_i(p), c_i, w_i, 1\} \quad \textbf{(33)}$$

which shows that the transformation from sell order to buy for a premium strategy requires that the buy premium be a function of the equilibrium price, p.[7] Accordingly, all orders, whether premium based or sales, can be transformed into an order for a notional purchase.

Let g_i be the difference between the prices and the maximum premium per dollar that order i is willing to pay as specified in order i's limit price, i.e.,

$$g \equiv C^*p - w, \quad \textbf{(34)}$$

where g be the n by 1 vector of g_i. Now, because of the nature of limit prices, we can express restrictions on x_i and v_i based on g_i: If $g_i > 0$, then the limit price for order i is below the equilibrium price for order i, and so the fill for order i must be 0. Thus, if $g_i > 0$, then $x_i = 0$ and $v_i = 0$. If $g_i < 0$, then the limit price for order i is above the price for order i, and so order i must be fully filled. Thus, if the order is specified in notional terms and $g_i < 0$, then $x_j = r_j$ and $v_j = u_j$. Further, if order i is in terms of notional and $g_i < 0$, then $x_i = r_i$ and $v_i = x_i\pi_{i.}$ If order i is expressed in terms of premium and $g_i < 0$, then $v_i = u_i$ and $x_i = u_i/\pi_i$. We summarize these constraints as follows:

(a) If $g_i > 0$, then $x_i = 0$ for both notional and premium orders.

(b) If $g_i < 0$, then $x_i = r_i$ for notional orders and $x_i = u_i/\pi_i$ for premium orders.

(c) If $g_i = 0$, then $0 \le x_i \le r_i$ for notional orders and $0 \le x_i \le u_i/\pi_i$ for premium orders.

7 To see that the buy premium must depend upon the equilibrium vector of state prices, assume a state space with two states, each with equilibrium state prices of 0.6 and 0.4 respectively. A sale of one dollar of premium for the first state is replicated with a purchase of 0.4/(1-0.4) premium of the second state.

B. Parimutuel Equilibrium Pricing Conditions

We now proceed to develop the mathematical formulation of the parimutuel equilibrium pricing conditions. After developing the necessary notation, we first prove that existence of a unique parimutuel equilibrium when all limit prices are equal to unity for buy orders and 0 for sell orders, i.e., all orders are market orders. We then provide a general parimutuel representation theorem which shows that all parimutuel equilibria in our microstructure are solutions to an eigenvalue problem. In the following section, we then relax the restriction on limit orders and prove the existence of a unique equilibrium for limit orders than can take on any value in the interval [0,1].

We begin by recalling from the previous section that

$$y \equiv C^T x. \quad \textbf{(35)}$$

We have here suppressed the possible dependence of $\underline{y}$ upon p which arises for orders based upon premium. Thus, y is a m by 1 vector where $\underline{y_j}$, the i-th element of y, is the executed payout per defined outcome j, j = 1, 2, …, m.

To avoid initial null states, we introduce the concept of *opening orders*. We assume that the auction sponsor (the entity corresponding to the derivatives dealer in our microstructure) will introduce into each state j a small amount of opening or initial premium denoted k_j. We now let t_j denote the *total executed premium* in state j. Thus,

$$t_j \equiv y_j p_j + k_j. \quad \textbf{(36)}$$

The executed premium in state j is the sum of the executed payout for state j times the price for state j plus the opening order amount for state j. Let t denote the total executed premium, which can be computed as the sum of the t_j's.

$$t \equiv \sum_{j=1}^{m} t_j = \sum_{j=1}^{m} (y_j p_j + k_j). \quad \textbf{(37)}$$

The definition of parimutuel state prices requires that the ratio of state prices equal the ratio of the executed premium amounts in each state, i.e.,

$$\frac{t_j}{t_k} = \frac{p_j}{p_k}. \quad \textbf{(38)}$$

Substituting for the definition of t_j and t_k yields

$$\frac{t_j}{t_k} = \frac{p_j y_j + k_j}{p_k y_k + k_k} = \frac{p_j}{p_k}. \quad \textbf{(39)}$$

Eliminating the denominators of this previous equality and summing over j yields

$$\sum_{j=1}^{m} p_j(p_k y_k + k_k) = \sum_{j=1}^{m} p_k(p_j y_j + k_j). \quad \textbf{(40)}$$

Substituting for t into the above equation yields the following

$$(p_k y_k + k_k)\left(\sum_{j=1}^{m} p_j\right) = p_k t. \quad \textbf{(41)}$$

By the assumption that the state prices sum to unity yields the following equation

$$(p_k y_k + k_k) = p_k t, \quad \textbf{(42)}$$

or rearranging terms

$$p_k = \frac{k_k}{t - y_k}. \quad \textbf{(43)}$$

This equation states that the state price or probability is a function of

(1) the premium amount of the opening order premium for state k, k_k;

(2) the total amount of premium executed in the auction t; and

(3) the total amount of executed notional in equilibrium for state k, y_k.

We can now formulate the following proposition regarding parimutuel equilibrium.

Proposition 5*: Given demands for orders with limit prices equal to unity for buy orders (and 0 for sell orders), there exists a unique parimutuel equilibrium.*

Proof: See Appendix II**.**

We are know in a position to formulate the following theorem regarding the parimutuel market microstructure:

Parimutuel Representation Theorem*: All parimutuel equilibria are solutions to the following eigenvalue problem:*

$$H(p)p = tp$$

$$x_i = \frac{u_i}{\pi_i} \quad \textit{premium orders}$$

$$t = \max_i(\lambda_i(H))$$

$$|p| \textit{ is eigenvector of } t$$

Proof*:* Define the matrix *H*, which has m rows and m columns where m is the number of defined contingent states in the parimutuel auction, as follows

$$H \equiv \begin{bmatrix} y_1 + k_1 & k_1 & k_1 & \cdots & k_1 \\ k_2 & y_2 + k_2 & k_2 & \cdots & k_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \\ k_m & k_m & k_m & \cdots & y_m + k_m \end{bmatrix} \tag{44}$$

H is a square matrix with each diagonal entry of H is equal to $y_j + k_j$. The off-diagonal entries for row j are equal to k_j for j = 1, 2, …, m. Recall that p is the vector of length m whose j-th element is p_j. Note that

$$Hp = \begin{bmatrix} y_1 p_1 + k_1 p_1 + k_1 p_2 + k_1 p_3 + ... + k_1 p_m \\ k_2 p_1 + y_2 p_2 + k_2 p_2 + k_2 p_3 + ... + k_2 p_m \\ k_3 p_1 + k_3 p_2 + y_3 p_3 + k_3 p_3 + ... + k_3 p_m \\ ... \\ k_m p_1 + k_m p_2 + k_m p_3 + ... + y_m p_m + k_m p_m \end{bmatrix} = \begin{bmatrix} y_1 p_1 + k_1 \left(\sum_{j=1}^{m} p_j \right) \\ y_2 p_2 + k_2 \left(\sum_{j=1}^{m} p_j \right) \\ y_3 p_3 + k_3 \left(\sum_{j=1}^{m} p_j \right) \\ ... \\ y_m p_m + k_m \left(\sum_{j=1}^{m} p_j \right) \end{bmatrix}. \tag{45}$$

Since the p_j's sum to unity (Eqn. 25) we can write

$$\text{H p} = \begin{bmatrix} y_1 p_1 + k_1 \\ y_2 p_2 + k_2 \\ y_3 p_3 + k_3 \\ ... \\ y_m p_m + k_m \end{bmatrix}. \tag{46}$$

Rewriting Eqn. 43, we have

$$p_j y_j + k_j = t p_j .$$

The left hand side of this expression is simply the j-th row of Hp. Thus we can write

$$Hp = tp , \tag{47}$$

which is the matrix equivalent to Eqn. 43.[8] The intuition for the eigenvalue representation is that a parimutuel pricing vector must lie in the null space of the net risk, since in a parimutuel mechanism all claim payouts are funded by premiums paid. The net risk of the parimutuel mechanism is:

---

[8] Michael Overton of the Courant Institute of New York University first suggested to us that our parimutuel problem might have an interesting eigensystem representation.

$$H - tI \quad \textbf{(48)}$$

Thus, a pricing vector which lies in the null space of the net risk means that there exists a solution to

$$(H - tI)p = 0\,, \quad \textbf{(49)}$$

which is the eigenvalue result.

In the absence of premium-based orders, Eqn. 47 can be viewed exactly as an eigenvalue representation, where t is the maximum eigenvalue of the matrix H and where p is the normalized eigenvector associated with the eigenvalue t. Thus, in the presence of notional based orders, it is trivial to see that the parimutuel system has a unique fixed point equilibrium. However, in the presence of premium orders, the solution of the eigenvalue problem must be accomplished iteratively, because the vector y depends on p. To see this, for a fully filled premium order we note that

$$x_i = u_i/\pi_I. \quad \textbf{(50)}$$

By the definition of π_i (Eqn. 26)

$$\pi_i = c_i^T\ p.$$

Since y is a linear combination of the x's, then y depends on p and so H depends on p. To make this dependence explicit we can write H as a function of p as H(p), and thus

$$H(p)p = tp\,. \quad \textbf{(51)}$$

In this general case, Eqn. 51 is not the standard eigenvalue/eigenvector representation. However, since the matrix H is convex in x and x is convex in p, we prove in the next section where we demonstrate the existence of a limit order book equilibrium, that there exists a unique equilibrium in the presence of premium-based orders.

C. <u>Parimutuel Limit Order Book Equilibrium</u>

We now introduce limit orders into the parimutuel equilibrium calculations. Limit orders are an important feature of the parimutuel microstructure under discussion. Traditional parimutuel wagering methods do not allow for either notional trading, limit orders, or bundling across risky states. These deficits render the raw parimutuel structure used for wagering less than optimal for use in the capital markets. As previously mentioned, options, futures, and other derivatives contracts are based upon notional contract size, rather than the amount to be invested in such contracts. Furthermore, parimutuel wagering markets expose participants to an excessive amount of transaction risk, as all wagers are executed at prices which vary throughout the auction period and are not known until all wagers have been made. In the capital markets, it is customary to use the device of limit prices to limit transaction risk by which participants can assure themselves that their orders are executed only if the market price is more favorable than their indicated limit price. Finally, parimutuel wagering is normally conducted in an ad hoc manner in which liquidity which could be aggregated within the same state space is fragmented into different "pools." For example, wagers on bets for a horse to win are held in a parimutuel pool which is separate from wagers on a horse to "place." This means that not only can there be arbitrage opportunities across the separate pools, the pricing within each pool is less efficient due to the disaggregation of liquidity. A viable parimutuel microstructure for the capital market should aggregate all liquidity within a state space, effectively allowing for the no-arbitrage bundling of any type of contingent claim from the fundamental state claims.

In the previous section, we have shown how a unique parimutuel equilibrium exists, for limit prices restricted to be equal to one for buy orders and 0 for sell orders. In this section, we prove the existence of a unique parimutuel price equilibrium for limit orders with limit prices that can take on any value between zero and one.

1. Limit Order Book Equilibrium

We regard limit orders as particularly important within the context of our microstructure for two reasons. First, they allow mitigation of execution risk owing to changing contingent claim prices during the auction period. In parimutuel wagering, an early bettor subjects himself to the risk that the final odds are lower than when the bet was placed. In our microstructure, we allow traders to control the execution price, effectively substituting a probability of execution at the limit price or better for the continuous change in odds faced by a parimutuel bettor. Second, limit orders are a familiar order execution mechanism in the capital markets which we believe should be incorporated into any viable and practicable microstructure for contingent claims.

With the introduction of limit orders comes the requirement of specifying an objective function for determining, subject to the satisfaction of the limit price constraints, which orders are executed in equilibrium. We choose to maximize the total volume of notional orders that can be executed subject to the limit price constraints. We do this for two reasons. First, we take as our definition of "liquidity" the maximum amount of notional value that can be accommodated in the auction subject to limit price constraints. Thus, the choice of objective function reflects the definition of liquidity which we are trying to maximize. Second, it is anticipated that the sponsor of the auction will earn transaction fee income as a percentage of notional for each order. Our choice of objective function therefore reflects choosing the set of orders that generate maximum fee income. The optimization problem can therefore be written in the following form:

$$
\begin{aligned}
& x^* = \underset{x}{argmax} \sum_{i=1}^{n} x_i \\
& \textit{subject to} \\
& (a)\, 0 < p_j < 1 \text{ for j} = 1, 2, \ldots, m \\
& (b) \sum_{j=1}^{m} p_j = 1 \\
& (c)\, p_j = \frac{k_j}{t - y_j} \text{ for j} = 1, 2, \ldots, m \\
& (d)\, g_i > 0 \rightarrow x_i = 0 \textit{ (for both notional and premium orders)} \\
& (e)\, g_i < 0 \rightarrow \begin{cases} x_i = r_i \textit{ (for notional orders)} \\ x_i = \dfrac{u_i}{\pi_i} \textit{(for premium orders)} \end{cases} \\
& (f)\, g_i = 0 \rightarrow \begin{cases} 0 \le x_i \le r_i \textit{ (for notional orders)} \\ 0 \le x_i \le \dfrac{u_i}{\pi_i} \textit{(for premium orders)} \end{cases}
\end{aligned} \qquad \textbf{(52)}
$$

We can simplify this problem by pre-processing the premium orders in order to convert them into notional orders. We first assume that all orders have been previously converted from sale orders to buy orders as discussed in IIA. We note that a premium order with limit price w_i can be approximated using notional-based limit orders by converting the premium-based order into a vector of notional-based orders. The approximating notional based vector contains a vector of notional requests and a vector of associated limit prices as follows:

$$
s_i = \{u_i, c_i, w_i, 1\} = \left\{ \left(\frac{u_i}{\pi(1)}, \frac{u_i}{\pi(2)}, \ldots, \frac{u_i}{w_i}\right), c_i, (\pi(1), \pi(2), \ldots, w_i), 1 \right\}, \qquad \textbf{(53)}
$$

where the vector

$$
(\pi(1), \pi(2), \ldots, w_i) \qquad \textbf{(54)}
$$

is understood to mean a fine partition of order prices up to and including the limit price of the premium-based order being approximated.

We can therefore confine our attention to notional-based buy orders. The limit order book optimization therefore simplifies to:

$$
\begin{aligned}
& x^* = \underset{x}{argmax} \sum_{i=1}^{n} x_i \\
& subject\ \ to \\
& (a)\, 0 < p_j < 1 \ \text{ for j} = 1, 2, \ldots, m \\
& (b) \sum_{j=1}^{m} p_j = 1 \\
& (c) p_j = \frac{k_j}{t - y_j} \ \text{ for j} = 1, 2, \ldots, m \\
& (d) g_i > 0 \rightarrow x_i = 0 \\
& (e) g_i < 0 \rightarrow x_i = r_i \\
& (f) g_i = 0 \rightarrow 0 \le x_i \le r_i
\end{aligned}
\qquad \textbf{(55)}
$$

The following proposition and proof demonstrate the existence of a unique price equilibrium.

Proposition 6*: The parimutuel limit order book problem of Eqn. 55 has a unique price equilibrium in the case of non-zero opening orders on each state.*

Proof*:* The proof is based upon fixed point continuation methods and is contained in Appendix II.

2. An Example of Limit Order Book Equilibrium

We provide a simple example of the solution of the parimutuel limit order book problem. In our example, we use the following input data:

m = 5 states

n = 8 orders

$$\mathbf{k} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} 100 \\ 200 \\ 300 \\ 400 \\ 200 \\ 350 \\ 100 \\ 150 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0.4 \\ 0.8 \\ 0.7 \\ 0.9 \\ 0.9 \\ 0.9 \\ 0.25 \\ 0.75 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The solution to the optimization problem is:

$$x^* = \begin{bmatrix} 100.000 \\ 109.560 \\ 0.000 \\ 0.000 \\ 8.571 \\ 99.011 \\ 100.000 \\ 0.000 \end{bmatrix}, \quad y^* = \begin{bmatrix} 208.571 \\ 207.582 \\ 107.582 \\ 217.142 \\ 208.571 \end{bmatrix}, \quad T^* = 218.571$$

$$p^* = \begin{bmatrix} 0.100003859 \\ 0.091003287 \\ 0.009009933 \\ 0.699979066 \\ 0.100003856 \end{bmatrix} \quad g^* = \begin{bmatrix} -0.208992855 \\ -0.0000170790 \\ 0.008988999 \\ 0.090990067 \\ -0.000003856 \\ -0.000003859 \\ -0.149996141 \\ 0.049992286 \end{bmatrix}$$

III. Parimutuel Microstructure: Arbitrage and Efficiency Considerations

We believe the parimutuel microstructure proposed and analyzed in Section II compares favorably to other microstructures that may be used for contingent claims trading. We think our parimutuel microstructure is superior to dealer-based and currently used exchange structures for a wide variety of risks, but especially for those risks which do not have tradable underlying securities or instruments. We organize our discussion of the benefits of our microstructure into the following six areas: (1) risk-neutrality; (2) the absence of arbitrage; (3) efficiency; (4) price uniqueness; (5) multilateral order matching and; (6) information production.

A. Risk Neutrality[9]

Parimutuel principles entail a self-funded auction of contingent claims: all premium collected, excluding transaction costs, is exactly sufficient to pay for all state contingent payouts. From a dealer perspective, the parimutuel microstructure will be preferable to standard OTC transactions for certain types of derivatives risks. For example, a dealer in fixed income derivatives will likely find the proposed parimutuel microstructure favorable for transacting options on the monthly announcement of non-farm payrolls since there is no underlying security or hedgable instrument.

The parimutuel microstructure we propose effects an arbitrage-free and riskless set of contingent claims prices and order executions. Effectively, the mechanism achieves what a dealer would need to do manually through hedging activity in an underlying instrument (where available) and through balancing risk by adjusting prices with trading counterparties to equilibrate net notional transactions across states. We think this simplicity and efficacy of the parimutuel microstructure as adapted to the capital markets is therefore a potentially useful complement to the traditional OTC dealer market structure, especially for types of risks which have no tradable underlying.

We also think that the proposed parimutuel microstructure is superior to conventional exchange-based continuous double auctions for some types of illiquid risks. For example, for a number of years the Chicago Board of Trade (CBOT) has offered options on insurance catastrophe losses as measured by indices published by the Property Claims Service (PCS). The microstructure used to transact these claims is a conventional continuous double auction, i.e., the same mechanism that is used to trade the highly liquid bond futures and options at the CBOT. While there are perhaps reasons why the PCS contracts have failed to attract liquidity which are unrelated to market microstructure, see, e.g., Cummins and Mahul (2000), we believe that the conventional microstructure may be a significant impediment to liquidity, as we discuss further below.

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9 By "risk-neutrality" we mean that the parimutuel auction is self-funding in the sense that premium inputs equal state contingent outputs. We do not mean to suggest a connection to the continuous time options literature which is focused on risk-neutral pricing.

B. Arbitrage-free Claims

A parimutuel system is arbitrage-free in the sense that there exists a positive state price vector which excludes arbitrage. Following the standard definitions (see Ingersoll (1987), p. 57), we can define the returns tableau, Z, of a parimutuel state space as follows:

$$Z = C * diag(\pi)^{-1} \quad \textbf{(56)}$$

Now, it is well known that if there exists a state pricing vector p supporting the returns tableau such that:

$$Zp = 1 \quad \textbf{(57)}$$

then there exists no arbitrage possibilities in the sense that there exists no investment η across the states which solves either:[10]

$$\begin{aligned} & e^T \eta \le 0 \\ & Z^T \eta \ge 0 \text{ (one strictly)} \\ \text{or} \quad & e^T \eta < 0 \\ & Z^T \eta \ge 0 \end{aligned} \quad \textbf{(58)}$$

In the proposed parimutuel market microstructure, a definition is that all contingent claim prices are linear combinations of the state prices, i.e.,

$$\pi = Cp\,. \quad \textbf{(59)}$$

Multiplication of this definition by $diag(\pi)^{-1}$ establishes that there is a supporting state price vector and that no arbitrage is possible by construction of the parimutuel microstructure.

The claim bundling feature of our parimutuel microstructure by definition rules out arbitrage in the above-defined sense. A market for state contingent claims, even a call auction like our own microstructure, need not enforce the no-arbitrage condition explicitly. Namely, we can readily envision a contingent claims market for a state space which can be modeled without such explicit restrictions as follows:

---

10 See Ingersoll (1987), pp. 54-57 for the elementary proof.

$$
\begin{aligned}
&x^* = \underset{x}{argmax} \sum_{i=1}^{n} x_i \\
&subject\ \ to \\
&(d) g_i > 0 \rightarrow x_i = 0 \\
&(e) g_i < 0 \rightarrow x_i = r_i \\
&(f) g_i = 0 \rightarrow 0 \le x_i \le r_i
\end{aligned}
\qquad \textbf{(60)}
$$

which is Eqn. 55 without the parimutuel and no-arbitrage price restrictions. In such a market, presumably arbitrageurs would devote capital to ensuring that arbitrage would be excluded from the prices. Our parimutuel mechanism enforces the normalization of state prices and the absence of such arbitrage endogenously within the microstructure.

C. Efficiency of Parimutuel Price Discovery

The enforcement of the no arbitrage conditions leads naturally to the following welfare result on the efficiency of our parimutuel microstructure compared to a model in which contingent claims are traded separately in a call auction over a state space (the "trading post" model).

***Proposition** 7: A parimutuel microstructure discovers prices for contingent claims such that the average order's standard deviation around fair value is less than a microstructure with separate call auction trading posts for each claim. The average order noise savings is equal to*

$$s = \alpha\bar{\sigma}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)T$$

where

s = savings due to parimutuel microstructure

T = total premium in system

$\bar{\sigma}$ *= average volatility of price error around the true price ("noise volatility")*

α = bid/offer spread assumed proportional to average noise volatility

Proof: See Appendix II.

We also note that the parimutuel mechanism that we propose has an additional efficiency gain over the traditional continuous market because of the time aggregation of orders provided by the call auction itself.[11]

There is suggestive empirical evidence supporting our simple efficiency result. Gabriel and Marsden (1990) and Gabriel and Marsden (1991) examine British betting on horses in which parimutuel and bookmakers make prices simultaneously. The bookmakers offer odds on wagers using the "starting price" odds convention, whereby a bookmaker takes a bet at odds formed by a consensus of bookmakers just before the race is run. Thus, both the parimutuel and starting price odds reflect odds just before the race is run. On the same sample of races, Gabriel and Marsden (1991, Table 1), find that parimutuel returns on the same races are about 28.7% higher, almost exactly the amount of efficiency owing to the parimutuel system we predict in Proposition 7.

D. Price Uniqueness

Our proposed parimutuel microstructure possesses a unique price equilibrium for a given set of opening orders and other orders for contingent claims. Not all microstructures of the type we propose possess unique equilibrium prices. Consider, in this regard, the following modified microstructure similar to the one we propose in Section II:

$$
\begin{aligned}
& x^* = \underset{x}{argmax} \sum_{i=1}^{n} x_i \\
& subject\ \ to \\
& (a)\, 0 < p_j < 1 \ \text{ for j} = 1, 2, \ldots, m \\
& (b) \sum_{j=1}^{m} p_j = 1 \\
& (c) (y^T p) e = y \\
& (d) g_i > 0 \rightarrow x_i = 0 \\
& (e) g_i < 0 \rightarrow x_i = r_i \\
& (f) g_i = 0 \rightarrow 0 \le x_i \le r_i
\end{aligned}
\qquad \textbf{(61)}
$$

This microstructure problem is otherwise identical with that of Eqn. 55 of Section II, except that the parimutuel constraint (c) in Eqn. 55 has been replaced with a weaker constraint in Eqn. 61. The constraint (c) in Eqn. 61 merely requires that the state

[11] See Economides and Schwartz (1995).

contingent payouts for each state be equal. This microstructure has some parimutuel features in the sense that elemental state claims are normalized, exhibit no arbitrage, and relative prices are equal to relative premium investments for each pair of states. Yet, there exists *no unique set of state prices* which satisfy Eqn. 61. To see this, we consider a state space with three states. Assume that there are 3 orders: a limit buy order for 300 notional covering state 1 at limit price of .3, a limit buy order for 200 notional covering state 2 at limit price of .4, and a limit buy order for 100 notional covering state 3 at limit price of .5. Clearly, any state probabilities satisfying

$$\begin{aligned}
&p_1 \le .3 \\
&p_2 \le .4 \\
&p_3 \le .5 \\
&p_1 + p_2 + p_3 = 1 \\
&p_1 y_1 + p_2 y_2 + p_3 y_3 = y_1 = y_2 = y_3
\end{aligned}$$

is a solution to (94), and there are obviously many such solutions, which will satisfy the risk neutrality constraint that all state payouts are equal. For example, one such solution is

$$\begin{aligned}
&p_1 = .25 \\
&p_2 = .25 \\
&p_3 = .5 \\
&p_1 + p_2 + p_3 = 1 \\
&p_1 y_1 + p_2 y_2 + p_3 y_3 = y_1 = y_2 = y_3 = 100
\end{aligned}$$

By contrast, the parimutuel microstructure we propose, embodied as the solution to Eqn. 55, possesses a unique set of state prices. In the simple example, under consideration, we assume that there exists opening orders on each state of one unit so that

$$k_1 = k_2 = k_3 = 1.$$

The unique solution is:

$$\begin{aligned}
&p_1 = .3 \\
&p_2 = .4 \\
&p_3 = .3 \\
&p_1 + p_2 + p_3 = 1 \\
&y_1 = y_3 = 100; y_2 = 100.8333 \\
&\sum_{i=1}^{3} p_i y_i + k_i = 103.333.
\end{aligned}$$

E. Multilateral Order-Matching

The parimutuel microstructure we propose is fundamentally a multilateral order-matching mechanism, by which we mean there exists no requirement of a discrete order match between a single buyer and a single seller. Rather, the order-matching mechanism is inherently "many-to-one" in the sense that any given contingent claim's payout is funded multilaterally by all of the other orders which are filled in equilibrium. We regard this feature as particularly important for claims for which there is no tradable underlying and for which there is not a natural demand for a continuous time market. For example, we regard our market microstructure to be of potential use to trade contingent claims on weather, economic statistic releases, corporate earnings releases, and mortgage prepayment speeds.

The character of our parimutuel microstructure is influenced greatly by the commitment of opening orders, $|k|$. For $|k| = 0$, the microstructure resembles a multilateral matching mechanism in which state prices are normalized, but are not necessarily unique. For $|k| \rightarrow \infty$, all orders which have limit prices better than the prices reflected in the opening orders will execute, and will have no impact on the state prices. Thus, large $|k|$ will tend to resemble a dealer microstructure in the sense that the dealer may bear significant risk that the distribution reflected in the opening orders distribution will depart from the "true" distribution. We believe the parimutuel microstructure we propose will tend to be most attractive at small values of k. We define small such that

$$1{,}000 < \frac{\sum_{j=1}^{n} x_j}{\sum_{i=1}^{m} k_i} < 10{,}000,$$

i.e., that the ratio of total notional volume filled in equilibrium to the total amount of opening orders is greater than 1,000 and less than 10,000.

F. Information Production

Our parimutuel microstructure discovers state prices through a state space partition of an underlying probability distribution. It therefore discovers the probability density function implied by actual trading activity in a transparent and natural way. We think the implied density produced in our microstructure will be an important and high quality informational externality to the market. The quality of the implied density will be high since the density itself is being traded "piece by piece" in our microstructure. The density discovered on our microstructure is always enforced to be a probability state space by design. Continuous time options markets, by contrast, produce asynchronous option prices at strikes which have varying liquidity and price noise. As a consequence, the traditional techniques used to extract implied density functions from continuous options data tend to produce very poor information due to data limitations and large noise in continuous time options prices (see Breeden and Litzenberger (1978)).

IV. <u>Conclusion</u>

A parimutuel market microstructure for contingent claims trading has been proposed and analyzed. A parimutuel microstructure is a call auction market with special equilibrium pricing conditions on the relative prices of contingent claims. We have shown that a parimutuel contingent claims mechanism is quite general, and has its roots in the market games literature.

We have shown how a market microstructure incorporating parimutuel principles for contingent claims trading which allows for notional transactions, limit orders, and bundling of claims across states can be constructed. We have proven the existence of a price equilibrium for such a market and suggest an algorithm for computing the equilibrium.

We believe that for a broad class of contingent claims, that the parimutuel microstructure developed in this paper offers many advantages over the dominant dealer and exchange continuous time mechanisms. First, the parimutuel mechanism does not require a discrete order match between two counterparties. Instead, orders are executed multilaterally. All executed order premium is used to fund all of the contingent in-the-money options, i.e., the payouts. Second, we believe the transparent and straightforward pricing mechanism will be attractive to market participants. We believe that the success of the parimutuel mechanism in the wagering markets can, with the modifications which we propose, be carried over into the capital markets. Third, we believe that the risk neutral and self-hedging nature of the parimutuel mechanism, from the perspective of the broker/dealer or other entity which hosts the auction, offers a superior tradeoff between the risk of derivatives dealing and the compensation for providing liquidity for contingent claims. We believe that the parimutuel microstructure may in fact avoid altogether some of risks inherent in derivatives market-making that periodically result in well-publicized disastrous outcomes. Finally, we believe that the parimutuel microstructure is ideally suited for completing some markets where there currently is an absence of liquidity, such as contingent claims on mortgage prepayment speeds, corporate earnings, weather, and economic statistics.

The potential for a new class of derivatives markets based upon our parimutuel microstructure is large. Already, some of the major financial institutions, including JP Morgan Chase and Deutsche Bank have committed to conducting parimutuel auctions on US, European, and Japanese economic statistic releases. We believe that our microstructure could also be used effectively in existing derivatives markets due to its efficiency in aggregating liquidity, both over time and across derivatives instruments.

Appendix I: Equilibrium Solution Techniques

The solution of the parimutuel limit order book optimization can pose some computational challenges. First, state prices depend upon the executed orders, i.e., the state prices are a function of the executed orders or $p(x)$. Second, the function $p(x)$ is nonlinear. Third, which orders are subject to execution are themselves a function of the prices, i.e., the function for order executions is $x(p(x))$. Thus, the optimization attempts to maximize the orders that can be executed subject to the following fixed point equilibrium:

$$x^* = x(p(x^*)) . \quad \textbf{(A1)}$$

Our preferred solution technique has two stages. In the first stage we construct a fixed point iteration using Newton's method as follows:

$$x(\kappa+1) = x(\kappa) - D_r(\kappa)^{-1} g(\kappa) , \quad \textbf{(A2)}$$

where the matrix $D_r(\kappa)$ is the fullest rank subspace of the Jacobian matrix of Eqn. A33, below. Executable orders are iterated until either the iteration step is tolerably converged or until the executable order has reached the order request amount. Orders which drop out of the iteration can be replaced with linearly dependent orders not part of the subspace matrix at the previous iteration.

When all orders have been converged under the fixed point iteration, the unique price equilibrium, $p^*(x)$, results. However, because of linear dependences in the matrix C, the executable orders may not yet be maximized. Our solution to obtain this maximum efficiently is to solve the following linear program taking $p^*(x)$ from the Newton's iteration as fixed:

$$\begin{aligned} & x^* = \underset{x}{argmax} \sum_{i=1}^{n} x_i \\ & subject \ \ to \\ & Hp^* = tp^* . \end{aligned} \quad \textbf{(A3)}$$

We have observed the computation of the inverse of the Jacobian subspace matrix $D_r(\kappa)^{-1}$ to be computationally intensive. We have achieved success using a fixed point method based upon dominant eigenvalue acceleration techniques. We use the fixed point iteration based upon

$$x(\kappa+1) = x(\kappa) - \beta(\kappa) g(x(\kappa)) \quad \textbf{(A4)}$$

and adjust the parameter $\beta(\kappa)$ based upon the value of the dominant eigenvalue of the Jacobian matrix $D(x(\kappa))$ at the current iteration. The smaller the dominant eigenvalue, the larger the step parameter $\beta(\kappa)$ can be accelerated to guarantee a locally convergent step.

Appendix II: Proofs of Propositions

Proof of Proposition 4:

We first show that the SSMG and a parimutuel market are payout-equivalent. We then show that the first order necessary conditions characterizing the Nash Equilibrium are identical in each market. With respect to payout equivalence, the proof of Proposition 2 shows that any strategy vector of premium bids and notional offers can be replicated using a bid strategy as follows. Namely, any strategy consisting of the following vector pair

$$\begin{aligned} x^i &= (x_1^i, x_2^i, \ldots, x_{m-1}^i, x_m^i), \\ b^i &= (b_1^i, b_2^i, \ldots, b_{m-1}^i, b_m^i), \end{aligned} \quad \textbf{(A5)}$$

which results in final wealth for agent i equal to

$$w_j^i(f) = w^i(o) - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} x_j^i p_j + \frac{b_j^i}{p_j} - x_j^i, \quad \forall \mathrm{j} = 1, \ldots, \mathrm{m}, \quad \textbf{(A6)}$$

can be replicated using a single vector strategy in bids as follows

$$b^i = (b_1^i + \sum_{j \neq 1} x_j^i p_1, b_2^i + \sum_{j \neq 2} x_j^i p_2, \ldots, b_{m-1}^i + \sum_{j \neq m-1} x_j^i p_{m-1}, b_m^i + \sum_{j \neq m} x_j^i p_m). \quad \textbf{(A7)}$$

Noting that

$$\sum_{j \neq 1} x_j^i p_1 + \sum_{j \neq 2} x_j^i p_2 + \ldots + \sum_{j \neq m-1} x_j^i p_{m-1} + \sum_{j \neq m} x_j^i p_m = \sum_{j=1}^{m} x_j^i (1 - p_j), \quad \textbf{(A8)}$$

the final state-contingent wealth for agent i owing to the replicated strategy is equal to

$$w_j^i(f) = w^i(o) - \sum_{j=1}^{m} b_j^i - \sum_{j=1}^{m} x_j^i (1-p_j) + \frac{b_j^i}{p_j} + \frac{p_j \sum_{k \neq j} x_k^i}{p_j}, \quad \forall \text{j} = 1, \ldots, \text{m},$$

$$w_j^i(f) = w^i(o) - \sum_{j=1}^{m} b_j^i - \sum_{j=1}^{m} x_j^i (1-p_j) + \frac{b_j^i}{p_j} + \sum_{j=1}^{m} x_j^i - x_j^i, \quad \forall \text{j} = 1, \ldots, \text{m}, \qquad \textbf{(A9)}$$

$$w_j^i(f) = w^i(o) - \sum_{j=1}^{m} b_j^i + \sum_{j=1}^{m} x_j^i p_j + \frac{b_j^i}{p_j} - x_j^i, \quad \forall \text{j} = 1, \ldots, \text{m}.$$

This shows that the final wealth from the replication strategy employing no offers is identical to the final wealth to the strategy employing offers, i.e., the SSMG for contingent claims and the parimutuel market are payout-equivalent.

We now show the first order necessary conditions of the parimutuel market are equivalent to those of the SSMG, as reported extensively in the market game literature. Since the entire strategy space can be obtained using bids, the optimization problem faced by agent i may be written as

$$\begin{aligned}
&\max_{b_j^i, j=1\ldots m} \sum_{j=1}^{m} q_j^i u(w_j^i(f)) = \sum_{j=1}^{m} \pi_j u(w^i(o) - \sum_{j=1}^{m} b_j^i + \frac{b_j^i}{p_j(b_j^i)}) \\
&\textit{subject to} \\
&\sum_{j=1}^{m} p_j(b_j^i) = 1, \\
&\sum_{j=1}^{m} q_j^i = 1, \qquad \textbf{(A10)} \\
&b_j^i \geq 0, \quad \forall \text{j} = 1, \ldots, \text{m}, \\
&\sum_{j=1}^{m} b_j^i \leq w^i(o),
\end{aligned}$$

where q_j^i denotes the subjective probability assessment of agent i for state j. Following Levin (1994), we find it more convenient to make the following change of variables in the optimization problem:

$$B_j^i = \sum_{k=1, k \neq i}^{n} b_j^k, \; B_j = \sum_{i=1}^{n} b_j^i, \; B = \sum_{j=1}^{m} B_j, \; B^i = \sum_{j=1}^{m} B_j^i. \qquad \textbf{(A11)}$$

Respectively, these new variables denote (1) the total bids for state j excepting agent i's bid; (2) the total bids for state j including agent i's bid; (3) the total bids for all states for all agents; and (4) the total bids for all the states excepting agent i's bids. Straightforward substitution of the new variables into the optimization problem yields:

$$\max_{B_j, B, j=1\ldots m} \sum_{j=1}^{m} q_j^i u(w_j^i(f)) = \sum_{j=1}^{m} \pi_j u(w^i(o) - B + B^i + \frac{B_j - B_j^i}{B_j} B)$$

$$\begin{aligned}
&\textit{subject to} \\
&\sum_{j=1}^{m} B_j = B, \\
&\sum_{j=1}^{m} q_j^i = 1, \\
&B_j \geq B_j^i, \quad \forall \mathrm{j} = 1, \ldots, \mathrm{m}, \\
&B \leq w^i(o) + B^i.
\end{aligned} \qquad \textbf{(A12)}$$

Straightforward differentiation of the associated Lagrangean yields the following first order necessary conditions for an interior optimum[12]

$$\frac{q_j^i u'(w_j^i(f))}{q_k^i u'(w_k^i(f))} = \frac{\frac{B_k^i B}{B_k^2}}{\frac{B_j^i B}{B_j^2}} = \frac{B_j^2}{B_k^2} \frac{B_k^i}{B_j^i}. \qquad \textbf{(A13)}$$

With the following definitions of the state contingent claim prices excluding the effect of agent i's strategy

$$p_j^i = \frac{B_j^i}{B^i} \qquad \textbf{(A14)}$$

and with the definition of the state price including i's strategy

$$p_j = \frac{B_j}{B}, \qquad \textbf{(A15)}$$

the first order conditions become

$$\frac{q_j^i u'(w_j^i(f))}{q_k^i u'(w_k^i(f))} = \frac{\frac{p_j^2}{p_j^i}}{\frac{p_k^2}{p_k^i}}, \qquad \textbf{(A16)}$$

which is identical to the optimality conditions derived by Peck, Shell, and Spear (1992, Proposition 2.4) for their implementation of the Shapley-Shubik commodities market game. Thus, a parimutuel market is both payout- and first-order-condition-equivalent to an SSMG market for contingent claims. This provides a connection between the

[12] By assuming an interior solution, we assume a positive bid for each state such that a "no-bid" strategy corresponds to a vanishingly small positive bid for a state.

extensive market game literature (Shapley and Shubik (1977), Peck, et al. (1992) and the smaller literature on parimutuel gambling (Levin (1994)).

Proof of Proposition 5[13]:

We provide a proof for notional orders, i.e., those with order amounts in terms of notional that are independent of equilibrium prices. The existence of equilibrium in the presence of orders based upon premium amount requests is proved in Section C. From Eqn. 43 and the assumption that the probabilities of the defined states must sum to one (Eqn. 25, again ignoring any interest rate considerations), the following m+1 equations may be solved to obtain the unique set of defined state prices (*p*'s) and the total executed premium

$$\text{(a) } p_j = \frac{k_j}{t - y_j}, j = 1,2,..,m, \quad \textbf{(A17)}$$

$$\text{(b) } \sum_{j=1}^{m} p_j = \sum_{j=1}^{m} \frac{k_j}{t - y_j} = 1.$$

Eqn. A17 contains m+1 unknowns and m+1 equations. The unknowns are the p_j's, i=1,2, …, m, and t, the total executed premium for all of the defined states.

We first solve for t. Using Eqns. 43 and the fact that p_j is greater than 0 and less than one, we conclude that

$$0 < \frac{k_j}{t - y_j} < 1 \text{ for j = 1, 2, …, m.} \quad \textbf{(A18)}$$

This equation implies that

$$t > y_j + k_j \text{ for j = 1, 2, …, m.} \quad \textbf{(A19)}$$

Thus,

$$t > \max(y_j + k_j) \text{ for j = 1, 2, …, m.} \quad \textbf{(A20)}$$

So a lower bound for t is equal to

$$t_{lower} = \max(y_j + k_j). \quad \textbf{(A21)}$$

where the maximum is taken over j = 1, 2, …, m.

[13] We would like to thank Ken Baron of Longitude who contributed to this proof.

Next, we derive an upper bound for t. Using the definition for t (Eqn. 37) and t_j (Eqn. 36),

$$t = \sum_{j=1}^{m} t_j = \sum_{j=1}^{m} k_j + \sum_{j=1}^{m} p_j y_j. \qquad \textbf{(A22)}$$

Letting $y_{(m)}$ be the maximum value of the y's,

$$t = \sum_{j=1}^{m} k_j + \sum_{j=1}^{m} p_j y_j \le \sum_{j=1}^{m} k_i + \sum_{j=1}^{m} p_j y_{(m)} = \sum_{j=1}^{m} k_j + y_{(m)} \sum_{j=1}^{m} p_j = \sum_{j=1}^{m} k_j + y_{(m)}. \qquad \textbf{(A23)}$$

Thus, the upper bound for t is equal to

$$t_{upper} = \sum_{j=1}^{m} k_j + y_{(m)} = \max(y_j) + \sum_{j=1}^{m} k_j. \qquad \textbf{(A24)}$$

The solution for the total premium in the defined outcomes therefore lies in the range $t \in (t_{lower}, t_{upper}]$ or

$$\max(y_j + k_j) < t \le \max(y_j) + \sum_{j=1}^{m} k_j. \qquad \textbf{(A25)}$$

Let the function f be defined as

$$f(t) = \sum_{j=1}^{m} \frac{k_j}{t - y_j} - 1. \qquad \textbf{(A26)}$$

Note that

$$f(t_{lower}) > 0, \qquad \textbf{(A27)}$$
$$f(t_{upper}) < 0.$$

Now, over the range $t \in (t_{lower}, t_{upper}]$, we can check that $f(t)$ is differentiable and strictly monotonically decreasing. Thus, we conclude that there is a unique t in the range such that

$$f(t) = 0. \qquad \textbf{(A28)}$$

Thus, t is uniquely determined from the y's and therefore the demands for orders which proves the proposition.

Once t is known, we can compute the vector p from Eqn. 43, since the k_j's are known. We now show how we can solve iteratively for t using the y's. Using Eqn. A26 we can write that

$$f'(t) = \frac{df}{dt} = -\sum_{i=1}^{m} \frac{k_i}{(\mathrm{t - y_j})^2}. \quad \textbf{(A29)}$$

Thus, for t take for an initial guess

$$t^0 = t_{lower}.$$

For the κ+1st guess, use

$$\mathrm{t}^{\kappa+1} = \mathrm{t}^{\kappa} - \frac{f(t^{\kappa})}{f'(t^{\kappa})}. \quad \textbf{(A30)}$$

The solution for $f(t) = 0$ over the interval $(t_{lower}, t_{upper}]$ can therefore be obtained using Newton's iteration. Once the solution is obtained, the value of t can be substituted into each of the m equations in Eqn. A17 to solve for the p_j.

Proof of Proposition 6:

We show that there exists a fixed point iteration sequence leading to a unique set of prices which solves the optimization problem. To prove the existence and convergence to a unique price equilibrium, consider the following iterative mapping

$$F(x) = x - \beta * g(x). \quad \textbf{(A31)}$$

Eqn. A31 can be proved to be contraction mapping which for a step size β independent of x will globally converge to a unique equilibrium, i.e., it can be proven that Eqn. A31 has a unique fixed point of the form

$$F(x^*) = x^*. \quad \textbf{(A32)}$$

To first show that $F(x)$ is a contraction mapping, matrix differentiation of Eqn. A31 yields:

$$\begin{aligned} &\frac{dF(x)}{dx} = I - \beta * D(x) \\ &where \\ &D(x) = B * A * Z^{-1} * B^T \\ &A_{i,j} = \begin{cases} p_i * (1 - p_i), i = j \\ -p_i * p_j, i \neq j \end{cases} \\ &Z_{i,j} = \begin{cases} T - y_i + p_i * y_i, i = j \\ p_j * y_i, i \neq j \end{cases} \end{aligned} \quad \textbf{(A33)}$$

The matrix $D(x)$ of Eqn. A33 is the matrix of order price first derivatives (i.e., the order price Jacobian). By well-known principles, Eqn. A33 can be shown to be a contraction if the following condition holds

$$\left|\frac{dF(x)}{dx}\right| < 1, \qquad \textbf{(A34)}$$

which is the case if the following condition holds

$$\begin{aligned} &\beta^* \rho(D) < 1, \\ &where \\ &\rho(D) = \max(\lambda_i(D)), i.e., the\ \ spectral\ \ radius\ \ of\ \ D. \end{aligned} \qquad \textbf{(A35)}$$

By the Gerschgorin's Circle Theorem, the eigenvalues of A are bounded between 0 and 1. The matrix Z^{-1} is a diagonally dominant matrix, all rows of which sum to $1/T$. Because of the diagonal dominance, the other eigenvalues of Z^{-1} are clustered around the diagonal elements of the matrix, and are approximately equal to p_i/k_i. The largest eigenvalue of Z^{-1} is therefore bounded above by $1/k_i$. The spectral radius of D is therefore bounded between 0 and linear combinations of $1/k_i$ as follows:

$$\rho(D) \le L,\ L = \frac{1}{\sum_{i=1}^{m} \frac{1}{k_i}}. \qquad \textbf{(A36)}$$

where the quantity L, a function of the opening order amounts, can be interpreted as the "liquidity capacitance" of the equilibrium (mathematically L is quite similar to the total capacitance of capacitors in series). The function $F(x)$ of Equation is therefore a contraction if

$$\beta < L. \qquad \textbf{(A37)}$$

Eqn. A37 states that a contraction to the unique price equilibrium can be guaranteed for contraction step sizes no larger than L, which is an increasing function of the opening orders in the auction.

The fixed point iteration of Eqn. A31 converges to x^*. Since $y^* = C^T x^*$, y^* can be used in Eqn. 43 to compute the fundamental state prices p^* and the total quantity of premium invested T^*. If there are linear dependencies in the C matrix, it may be possible to preserve p^* through a different allocation of the x's corresponding to the linearly dependent rows of *C*. For example, consider two orders, x_1 and x_2, which span the same states and have the same limit order price. Assume that $r_1 = 100$ and $r_2 = 100$ and that $x_1^* = x_2^* = 50$ from the fixed point iteration. Then clearly, it would be possible to set $x_1 = 100$ and $x_2 = 0$ without disturbing p^*. For example, different order priority rules may give execution precedence to the earlier submitted identical order. In any event, the fixed point iteration results in a unique *price* equilibrium, that is, unique in *p*. In our current model of the parimutuel limit order book, the priority rule is the optimization of the total notional orders subject to the optimal prices. At the optimal prices, the nonlinear program in Eqn. 55, becomes the following linear program:

$$\begin{aligned} & x^* = \underset{x}{argmax} \sum_{i=1}^{n} x_i \\ & subject\ \ to \\ & Hp^* = tp^* \end{aligned} \qquad \textbf{(A38)}$$

where p* solves the fixed point iteration.

***Proof of Proposition** 7*:[14]

Assume a market for m single state claims. We model the market price of these claims as

$$\begin{aligned} & \tilde{p}_j = \mu_j + \tilde{f}_j \\ & \tilde{f}_j \sim (0, \sigma_j^2) \end{aligned} \qquad \textbf{(A39)}$$

for j = 1, 2, …, m. In the parimutuel microstructure, the sum of the forward state prices are enforced to be one, or

$$\sum_{j=1}^{m} \tilde{p}_j = 1 . \qquad \textbf{(A40)}$$

Therefore,

$$Var\left(\sum_{j=1}^{m} \tilde{p}_j \right) = 0 . \qquad \textbf{(A41)}$$

Therefore,

$$Var\left(\sum_{j=1}^{m} \tilde{p}_j \right) = \sum_{j=1}^{m} \sigma_j^2 + \sum_{i \neq j} \sum_{i,j=1}^{m} \mathrm{cov}(\tilde{f}_i, \tilde{f}_j) = 0. \qquad \textbf{(A42)}$$

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[14] We would like to thank Ken Baron of Longitude for helpful discussions regarding this proof.

Next, let

$$\overline{\text{cov}} = \frac{1}{m(m-1)} \sum_{i \neq j} \sum_{i,j=1}^{m} \text{cov}(\tilde{f}_i, \tilde{f}_j),$$

$$\overline{\sigma^2} = \frac{1}{m} \sum_{j=1}^{m} \sigma_j^2 . \qquad \textbf{(A43)}$$

Substituting and rearranging terms yields:

$$\sum_{i \neq j} \sum_{i,j=1}^{m} \text{cov}(\tilde{f}_i, \tilde{f}_j) = -\sum_{j=1}^{m} \sigma_j^2$$

$$m(m-1)\overline{\text{cov}} = -m\overline{\sigma^2} \qquad \textbf{(A44)}$$

$$\overline{\text{cov}} = -\frac{\overline{\sigma^2}}{m-1}.$$

Now, we make a simplifying assumption of unit variances. This will not affect our analysis as we are interested in relative average noise between a parimutuel and trading-post microstructure. Thus, the last equation simplifies to

$$\overline{cov} = -\frac{1}{m-1} \qquad \textbf{(A45)}$$

We now analyze, using a simple tableau, the total variance of a contingent claim consisting of 1,2, …, m states in the parimutuel microstructure which imposes a covariance structure, and a non-parimutuel microstructure in which covariances are zero.

(A46)

Number of States	*Number of Orders*	*Variance of Order (PM)*	*Variance of Order (TM)*
1	$\binom{m}{1}$	$1\left(\frac{m-1}{m-1}\right)$	1
2	$\binom{m}{2}$	$2\left(\frac{m-2}{m-1}\right)$	2
3	$\binom{m}{3}$	$3\left(\frac{m-3}{m-1}\right)$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$\binom{m}{k-1}$	$k\left(\frac{m-k}{m-1}\right)$	k
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m-1	$\binom{m}{m-1}$	$\frac{m-1}{m-1}$	$m-1$
m	1	0	m

We now calculate the total variance (TV) of orders in a parimutuel (PM) microstructure and trading post microstructure (non-parimutuel) as follows:

$$TV_{PM} = \sum_{k=1}^{m} \binom{m}{k} \frac{k(m-k)}{m-1}$$

$$\sum_{k=1}^{m-1} \binom{m}{k} \frac{k(m-k)}{m-1}$$

$$= \frac{1}{m-1} \sum_{k=1}^{m-1} \frac{m!k(m-k)}{k!(m-k)!} \quad \textbf{(A47)}$$

$$= \frac{1}{m-1} \sum_{k=1}^{m-1} \frac{m(m-1)(m-2)!}{(k-1)!(m-k-1)!}$$

$$= m \sum_{l=0}^{m-2} \binom{m-2}{l}$$

$$= m2^{m-2},$$

since $\sum_{l=0}^{m-2} \binom{m-2}{l} = 2^{m-2}$

Further note that

$$TV_{TP} = \sum_{k=1}^{m} \binom{m}{k} k$$

$$= \sum_{k=1}^{m} \frac{m!k}{k!(m-k!)}$$

$$= \sum_{k=1}^{m} \frac{m!}{(k-1)!(m-k)!}$$

$$= \sum_{k=1}^{m} \frac{m(m-1)!}{(k-1)!(m-k)!} \quad \textbf{(A48)}$$

$$= m \sum_{k=1}^{m} \binom{m-1}{k-1}$$

$$= m \sum_{l=0}^{m-1} \binom{m-1}{l}$$

$$= m2^{m-1}$$

since $\sum_{l=0}^{m-1} \binom{m-1}{l} = 2^{m-1}$. Hence the ratio of trading post average order noise to parimutuel order noise is

$$\frac{TV_{TP}}{TV_{PM}} = \frac{m2^{m-1}}{m2^{m-2}} = 2\,. \qquad \textbf{(A49)}$$

So the average order noise for a parimutuel system is half that for the non-parimutuel system.

Assume that average noise volatility is 10% of the price. If therefore there is 100 million USD in premium, 10 million USD is one standard deviation of noise around the true price. By the previous result, a parimutuel microstructure would have 7.07 million in noise (10 million divided by square root of 2). Therefore, if the average bid-offer spread in a non-parimutuel microstructure is proportional to the noise volatility of prices, the net efficiency of the parimutuel system can be written as:

$$s = \alpha\bar{\sigma}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)T \qquad \textbf{(A50)}$$

where

s = savings due to parimutuel system

T = total premium in system

α = bid/offer proportionality constant

which is *Proposition 7*.

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